

INTERNATIONAL ACCELERATOR SCHOOL

ON
SUPERCONDUCTING SCIENCE AND TECHNOLOGY FOR PARTICLE ACCELERATORS



SASKATOON, CANADA 10 TO 20 JULY 2023

LONGITUDINAL DYNAMICS

Frank Tecker
CERN, ATS-DO



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The CERN Accelerator School

IAS Superconductivity, Saskatoon, Canada, July 2023

Main High-Energy Frontier Collider Projects

Circular colliders:

- **HL-LHC** (CERN) – High-Luminosity upgrade of LHC
- **FCC** (CERN) (Future Circular Collider)
 - FCC-hh: 100 TeV proton-proton cms energy, ion operation possible
 - FCC-ee: 90-350 GeV e^+e^- collider as potential intermediate step
 - FCC-he: Lepton-hadron option
- **CEPC / SppC** (China) (Circular Electron-positron Collider/Super Proton-proton Collider)
 - CepC : e^+e^- 240GeV cms
 - SppC : pp 70TeV cms
- **Muon collider**: 3-10 (14) TeV cms energy

Linear colliders:

- **ILC** (International Linear Collider): e^+e^- , 250-500 GeV cms energy, SC technology
Japan considers hosting project
- **CLIC** (Compact Linear Collider): e^+e^- , 380GeV-3TeV cms energy, NC technology
CERN hosts collaboration

FCC-ee: RF challenge

double ring e^+e^- collider ~ 100 km, cms energies:

Z (90 GeV), **W** (160 GeV), **H** (240 GeV), $t\bar{t}$ (350 GeV)

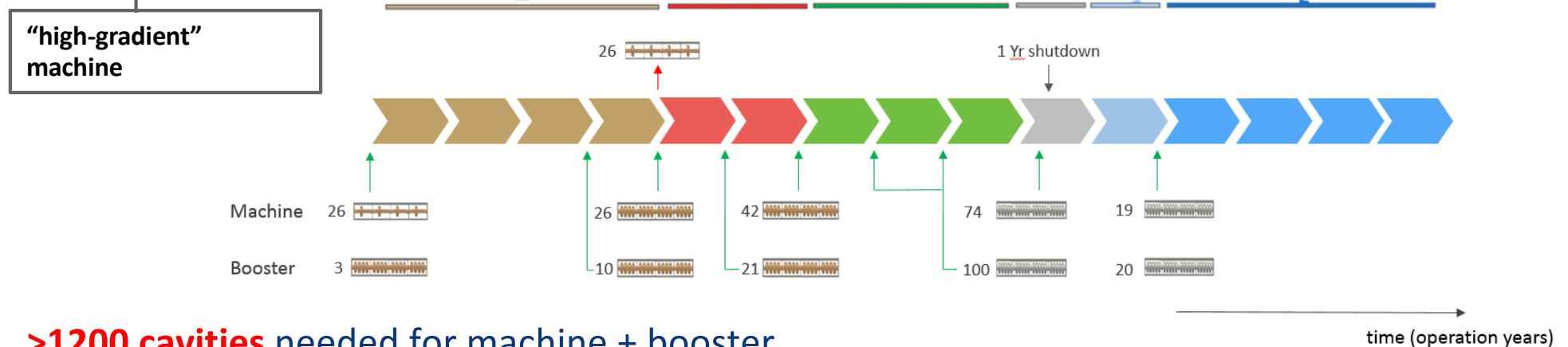
synchrotron radiation power **50 MW/beam** at all beam energies, scales with E^4

top-up injection requires booster synchrotron in collider tunnel

“Ampere-class” machine			
WP	V_{rf} [GV]	#bunches	I_{beam} [mA]
Z	0.1	16640	1390
W	0.44	2000	147
H	2.0	393	29
ttbar	10.9	48	5.4

three sets of RF cavities to cover all options for FCC-ee & booster:

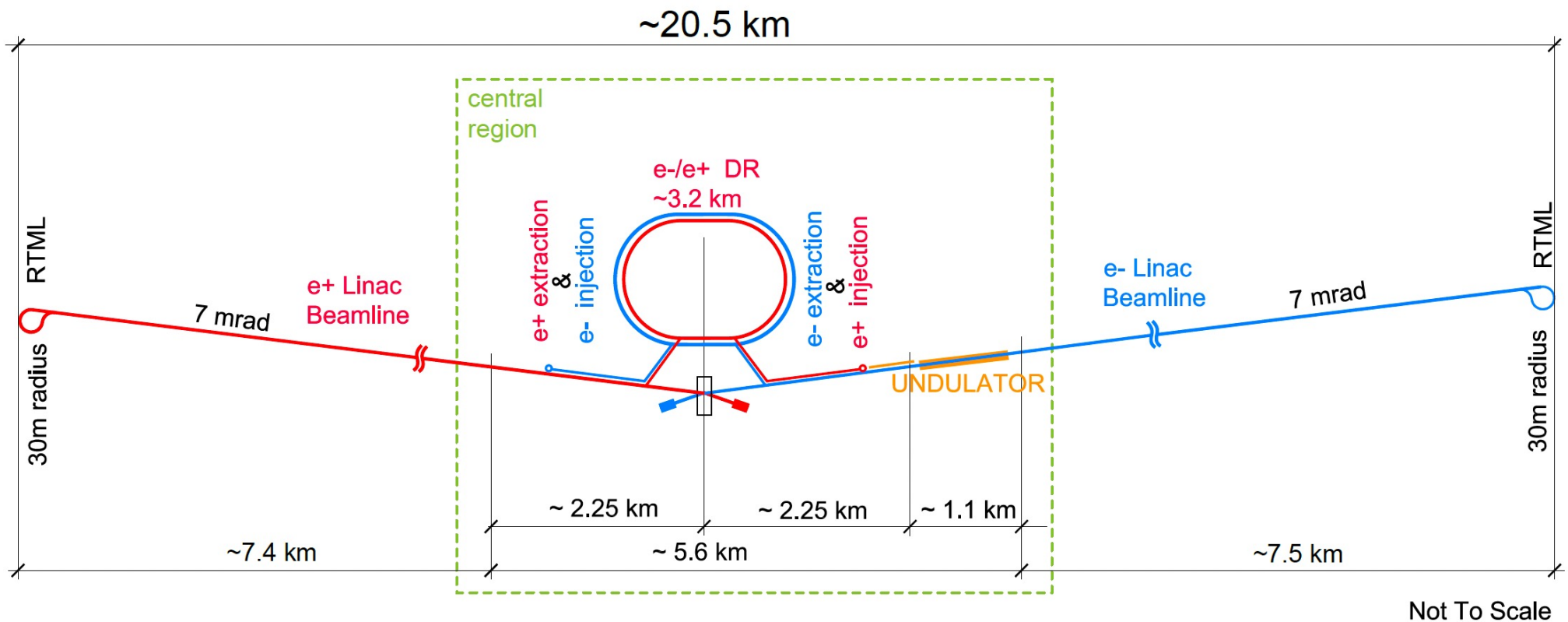
- high intensity (Z, FCC-hh): **400 MHz mono-cell cavities (4/cryom.)**
- higher energy (W, H, t): **400 MHz four-cell cavities (4/cryomodule)**
- ttbar machine complement: **800 MHz five-cell cavities (4/cryom.)**
- installation sequence comparable to LEP (≈ 30 CM/shutdown)



>1200 cavities needed for machine + booster

R&D aimed at improving performance & efficiency and reducing cost

Linear Collider: ILC



- 2 x 125 GeV linacs to produce nearly head-on e+e- collisions
2 x 250 GeV later
- Superconducting cavities with 31.5 MV/m gradient

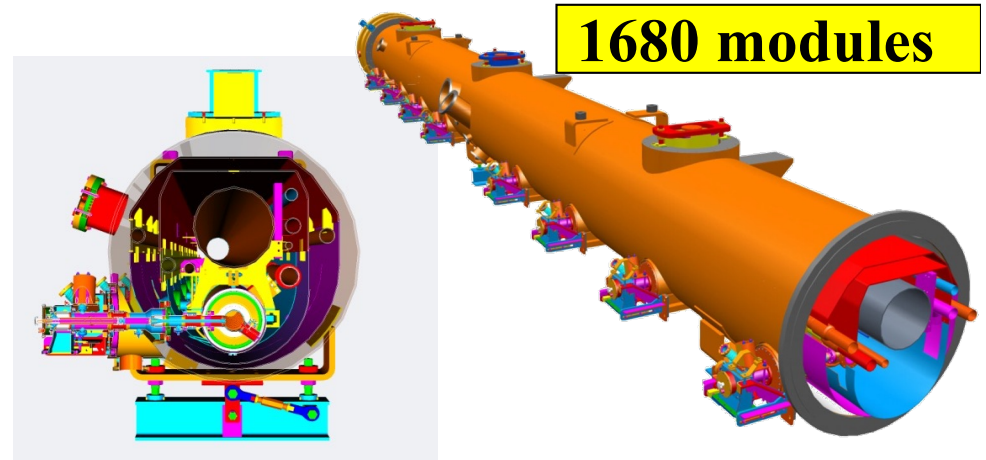


14560 cavities

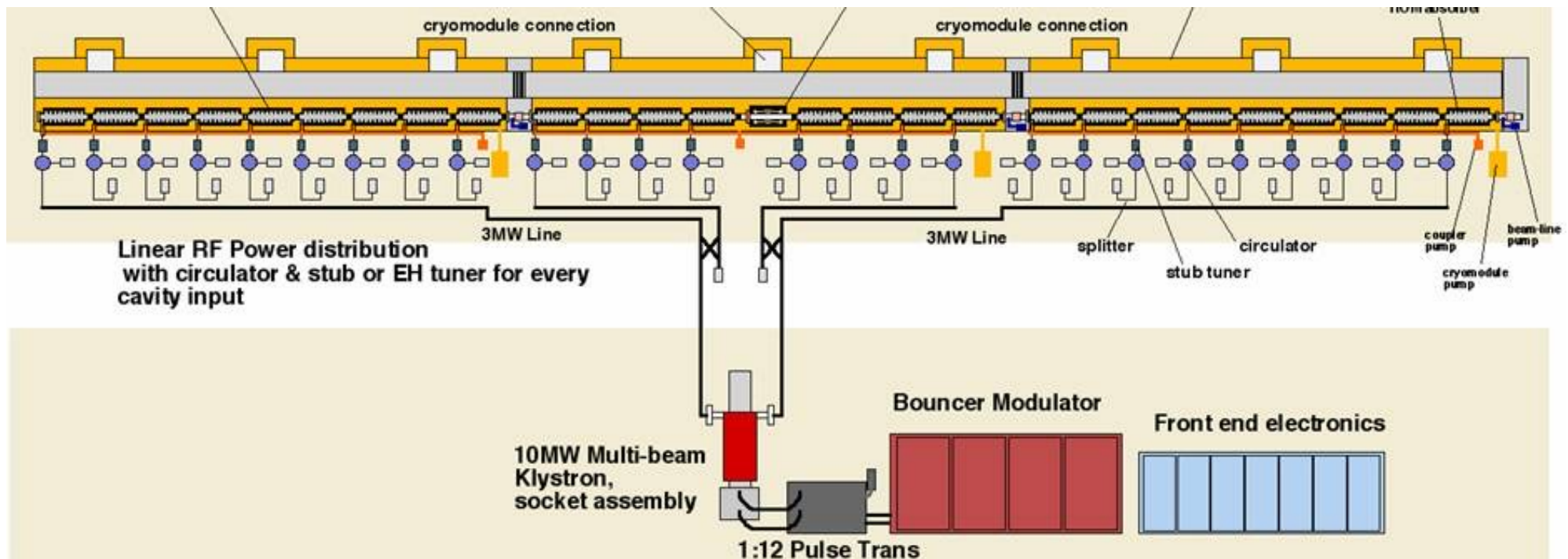
ILC Main Linac RF Unit

560 RF units each one composed of:

- 1 modulator
- 1 Multibeam klystron (10 MW, 1.6 ms)
- 3 Cryostats (9+8+9 = 26 cavities)
- 1 Quadrupole at the center



Total of 1680 cryomodules and 14 560 SC RF cavities



Summary of the 2 lectures:

- Acceleration methods
- Accelerating structures
- Linac: Phase Stability + Energy-Phase oscillations
- Circular accelerators: Cyclotron / Synchrotron
- Dispersion Effects in Synchrotron, Transition
- Stability and Longitudinal Phase Space Motion
- Hamiltonian
- Stationary Bucket
- Injection Matching
- Filamentation
- Bunch splitting

Particle types and acceleration

The accelerating system will depend upon the **evolution** of the **particle velocity**:

- **electrons** reach a **constant velocity** (~speed of light) at relatively low energy
- **heavy particles** reach a constant velocity only at very high energy
 - > need different types of resonators, optimized for different velocities
 - > the **revolution frequency will vary**, so the **RF frequency** will be **changing**
 - > magnetic field needs to follow the momentum increase

Particle rest mass m_0 :

electron 0.511 MeV

proton 938 MeV

^{239}U ~220000 MeV

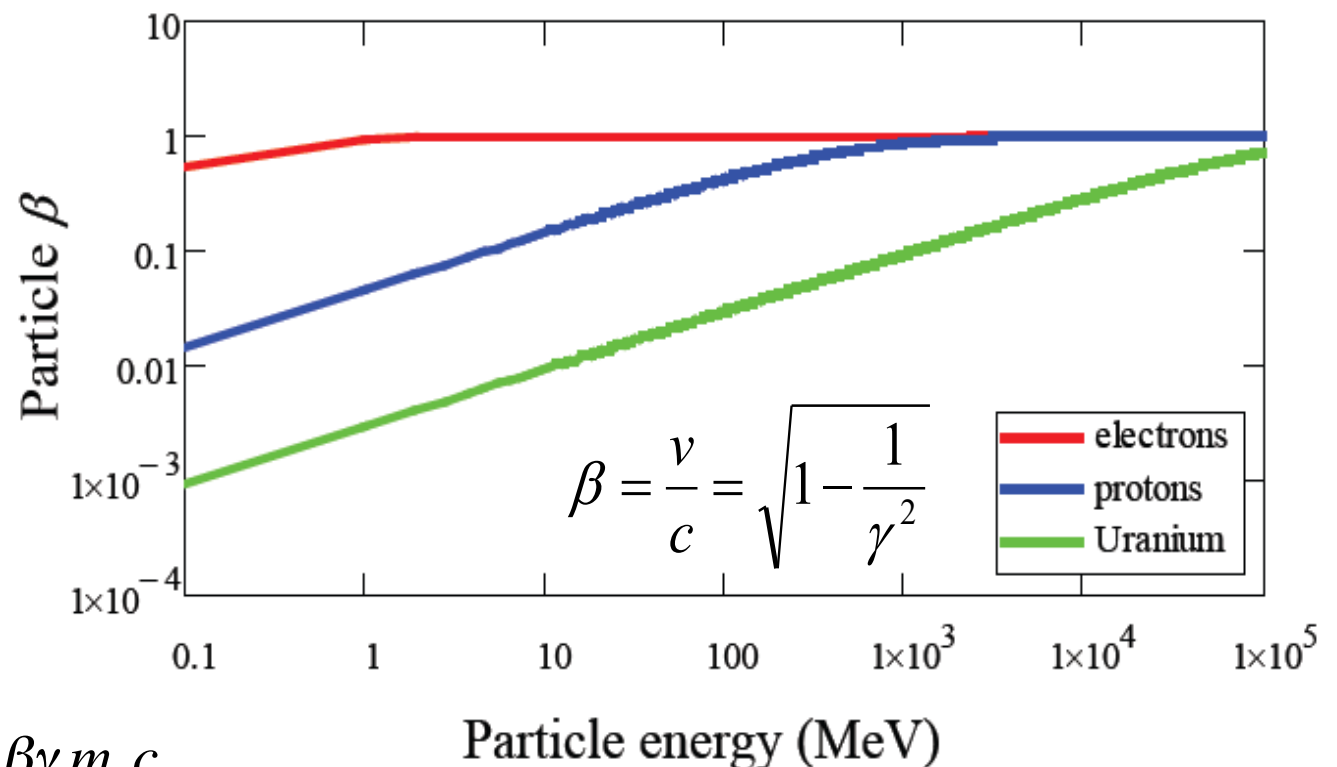
Total Energy: $E = \gamma m_0 c^2$

Relativistic
gamma factor:

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \beta^2}}$$

Momentum:

$$p = mv = \frac{E}{c^2} \beta c = \beta \frac{E}{c} = \beta \gamma m_0 c$$



Acceleration + Energy Gain

May the force
be with you!



To accelerate, we need a **force in the direction of motion!**

**Newton-Lorentz Force
on a charged particle:**

$$\vec{F} = \frac{d\vec{p}}{dt} = e \left(\vec{E} + \vec{v} \times \vec{B} \right)$$

2nd term always perpendicular
to motion \Rightarrow **no acceleration**

Hence, it is necessary to have an **electric field E**
(preferably) **along the direction of the initial momentum (z)**,
which changes the momentum p of the particle.

$$\frac{dp}{dt} = eE_z$$

In relativistic dynamics, total **energy E** and **momentum p** are **linked by**

$$E^2 = E_0^2 + p^2 c^2 \quad \Rightarrow \quad dE = v dp \quad \left(2E dE = 2c^2 p dp \Leftrightarrow dE = c^2 mv / E dp = v dp \right)$$

The rate of **energy gain per unit length** of acceleration (along z) is then:

$$\frac{dE}{dz} = v \frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

and the kinetic **energy gained** from the field along the z path is:

$$dW = dE = q E_z dz \quad \rightarrow \quad W = q \int E_z dz = qV \quad \begin{array}{l} - V \text{ is a potential} \\ - q \text{ the charge} \end{array}$$

Unit of Energy

Today's accelerators and future projects work/aim at the **TeV energy** range.

LHC: 7 TeV \rightarrow 14 TeV

CLIC: 3 TeV

FCC-hh: \sim 100 TeV

In fact, this energy unit comes from acceleration:

1 eV (electron Volt) is the energy that 1 elementary charge e (like one electron or proton) gains when it is accelerated in a potential (voltage) difference of 1 Volt.

Basic Unit: eV (electron Volt)

keV = 1000 eV = 10^3 eV

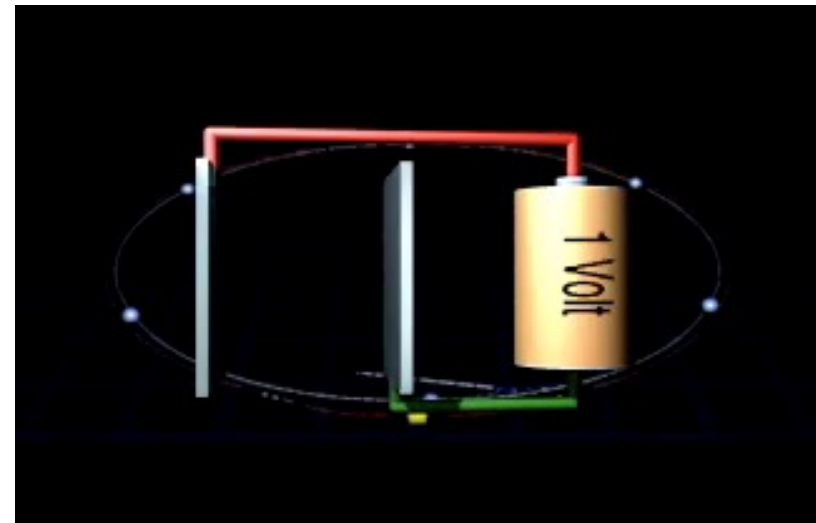
MeV = 10^6 eV

GeV = 10^9 eV

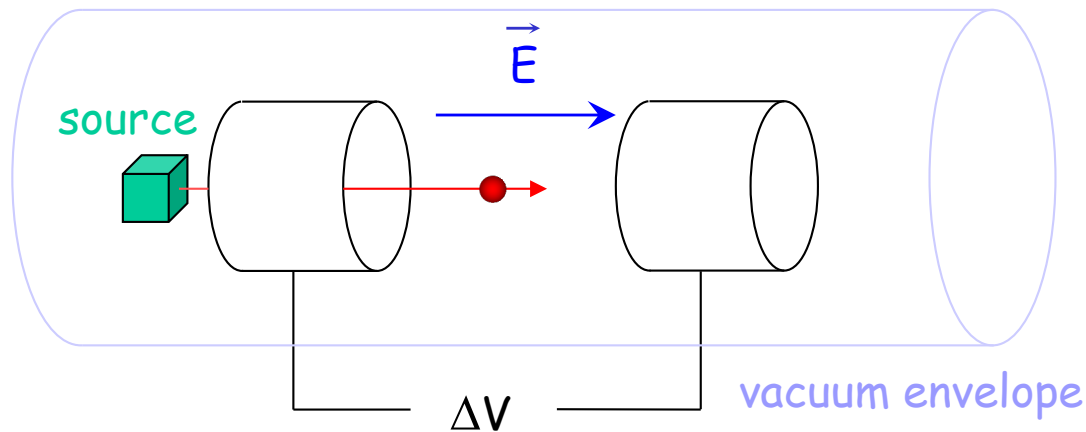
TeV = 10^{12} eV

LHC = \sim 450 Million km of batteries!!!

3x distance Earth-Sun



Electrostatic Acceleration



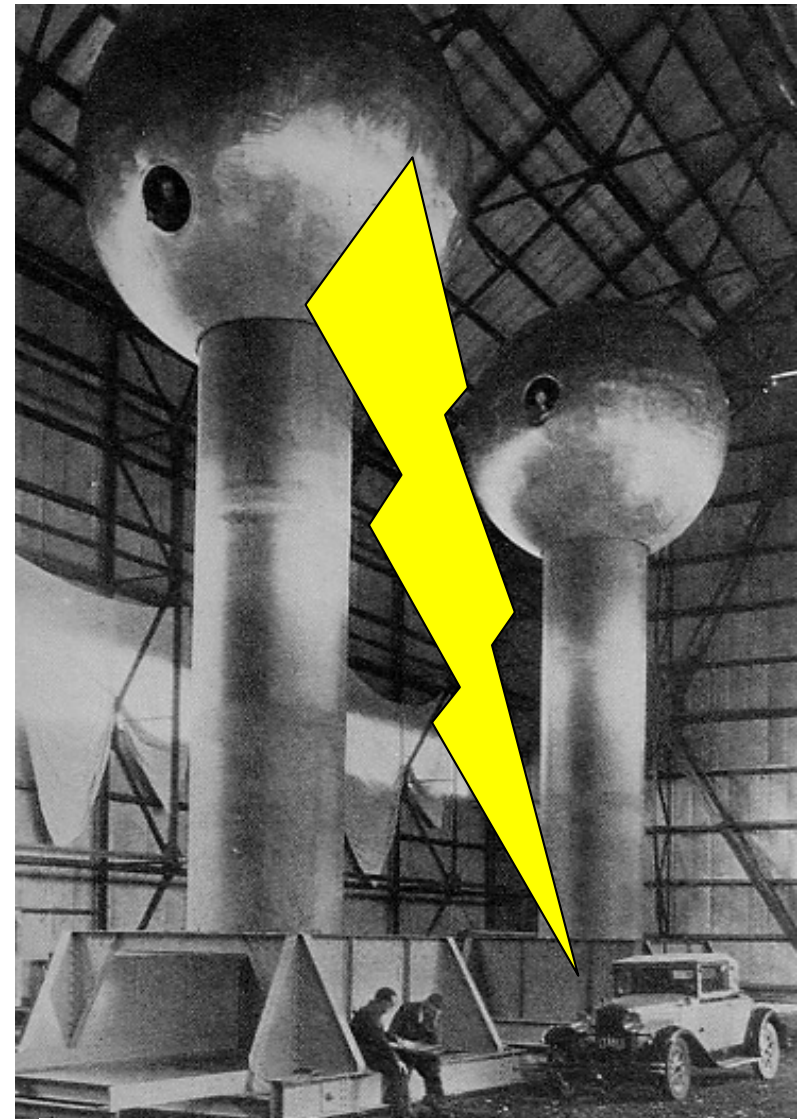
Electrostatic Field:

Force: $\vec{F} = \frac{d\vec{p}}{dt} = q \vec{E}$

Energy gain: $W = q \Delta V$

used for first stage of acceleration:
particle sources, electron guns,
x-ray tubes

Limitation: **insulation problems**
maximum high voltage (~ 10 MV)



Van-de-Graaff generator at MIT
<http://ark.cdlib.org/ark:/13030/ft5s200764/>

Methods of Acceleration: Time varying fields

The electrostatic field is limited by insulation,
the magnetic field does not accelerate.

Circular machine: DC acceleration impossible since $\oint \vec{E} \cdot d\vec{s} = 0$

From Maxwell's Equations: $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$

$$\vec{B} = \mu \vec{H} = \vec{\nabla} \times \vec{A} \quad \text{or} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

The electric field is derived from a scalar potential ϕ and a vector potential A
The time variation of the magnetic field H generates an electric field E

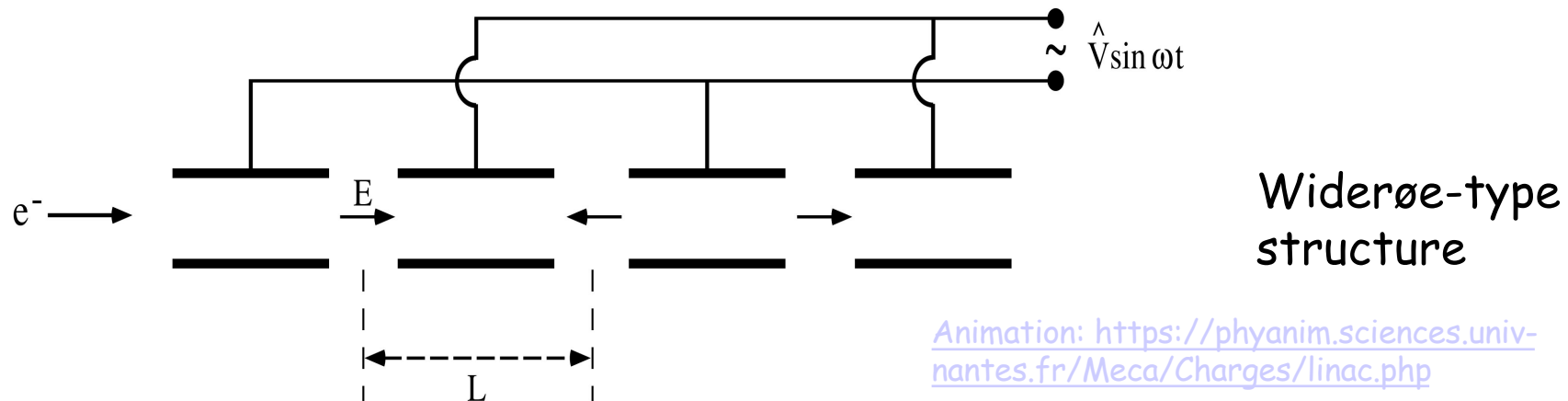
The solution: => time varying electric fields !

- 1) Induction (=> Betatron)
- 2) RF frequency fields

Consequence: We can only accelerate bunched beam!



Radio-Frequency (RF) Acceleration

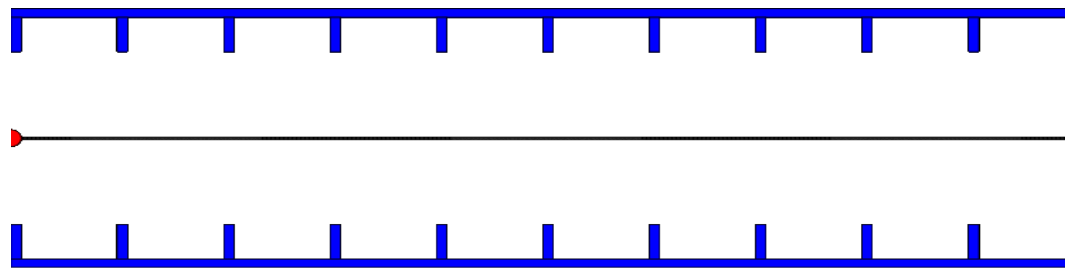


Cylindrical electrodes (**drift tubes**) separated by gaps and fed by a **RF generator**, as shown above, lead to an alternating electric field polarity

Synchronism condition

$$\longrightarrow L = v T/2$$

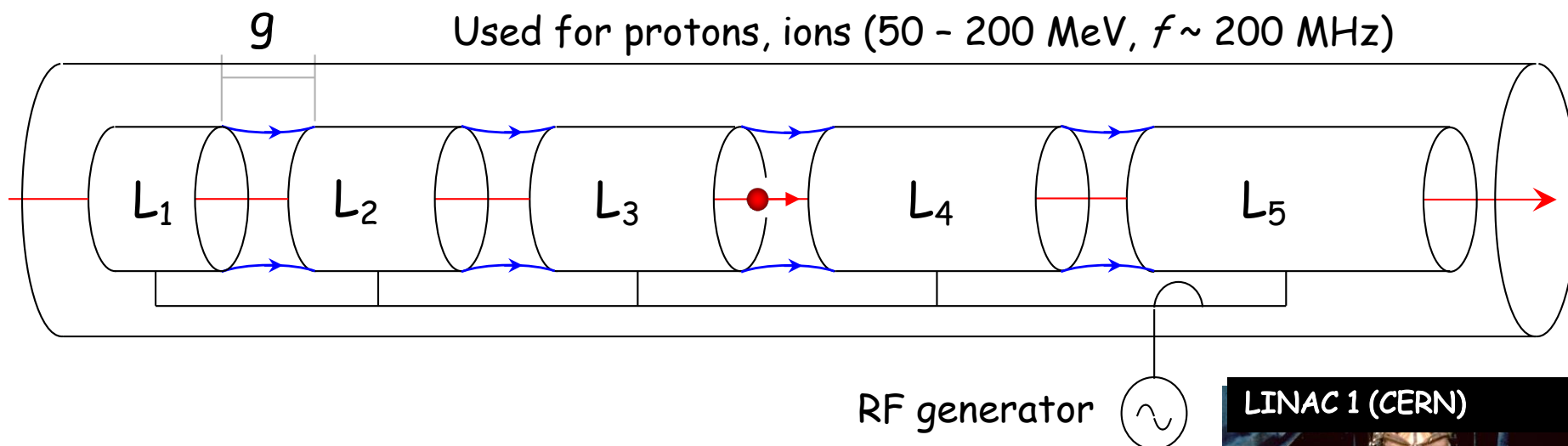
v = particle velocity
 T = RF period



Similar for standing wave cavity as shown (with $v \approx c$)

D.Schulte

RF acceleration: Alvarez Structure

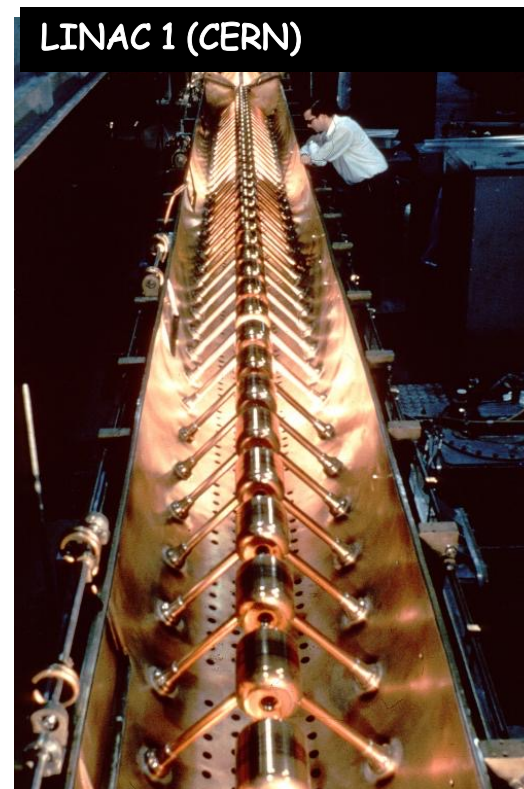


Synchronism condition ($g \ll L$)



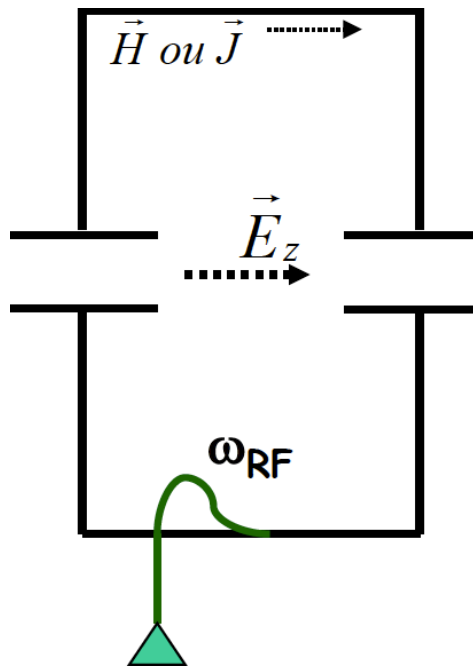
$$L = v_s T_{RF} = \beta_s \lambda_{RF}$$

$$\omega_{RF} = 2\pi f_{RF} = 2\pi \frac{v_s}{L}$$



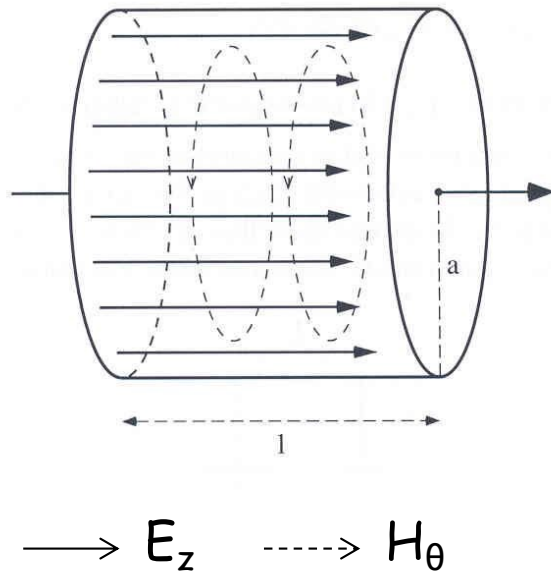
Resonant RF Cavities

- Considering RF acceleration, it is obvious that when particles get high velocities the drift spaces get longer and one loses on the efficiency.
=> The solution consists of using a **higher operating frequency**.
- The **power lost** by radiation, due to circulating currents on the electrodes, is **proportional to the RF frequency**.
=> The solution consists of **enclosing the system in a cavity** which resonant frequency matches the RF generator frequency.



- The electromagnetic power is now constrained in the resonant volume
- Each such cavity can be independently powered from the RF generator
- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)

The Pill Box Cavity



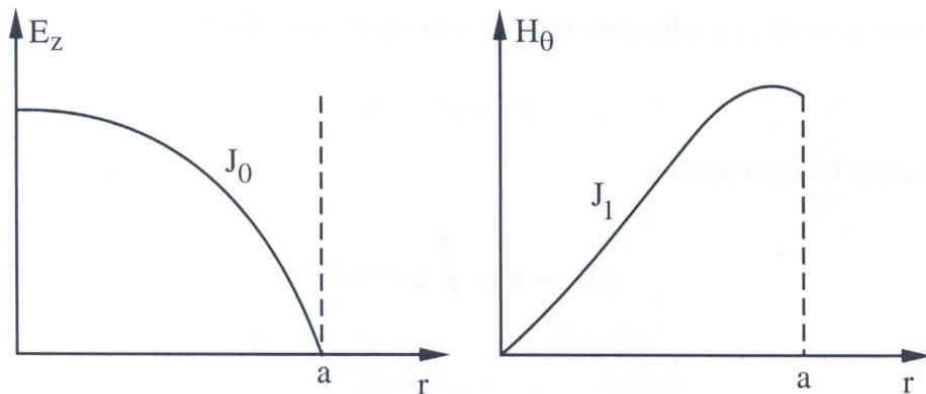
From Maxwell's equations one can derive the **wave equations**:

$$\nabla^2 A - \epsilon_0 \mu_0 \frac{\partial^2 A}{\partial t^2} = 0 \quad (A = E \text{ or } H)$$

Solutions for E and H are **oscillating modes**, at **discrete frequencies**, of types TM_{xyz} (transverse magnetic) or TE_{xyz} (transverse electric).

Indices linked to the **number of field knots** in polar co-ordinates φ , r and z .

For $k \ll 2a$ the most simple mode, TM_{010} , has the lowest frequency, and has only two field components:



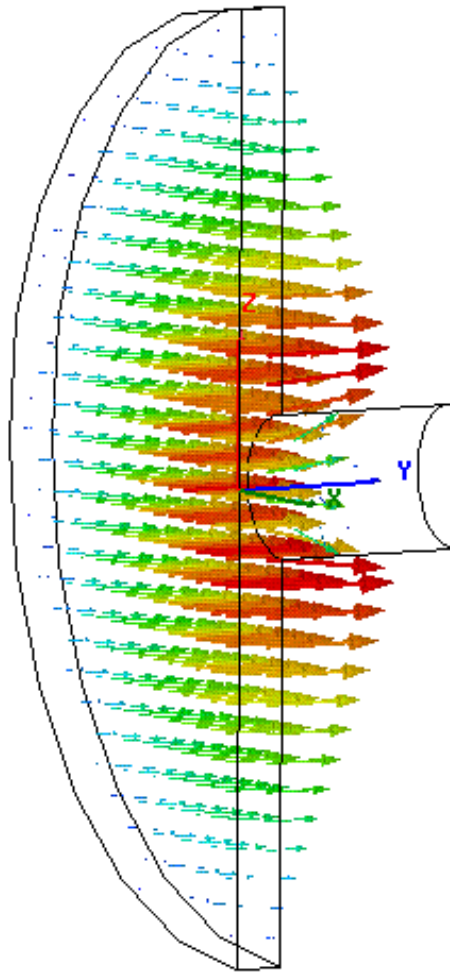
$$E_z = J_0(kr) e^{i\omega t}$$

$$H_\theta = -\frac{i}{Z_0} J_1(kr) e^{i\omega t}$$

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad \lambda = 2.62a \quad Z_0 = 377\Omega$$

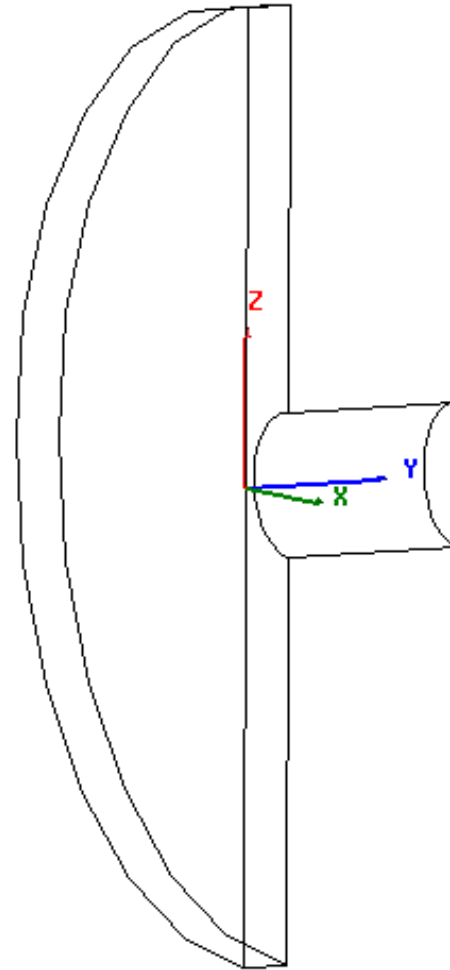
The Pill Box Cavity

One needs a hole for the beam pipe - circular waveguide below cutoff



electric field

TM_{010} -mode
(only 1/4 shown)



magnetic field

Transit time factor

The accelerating **field varies during the passage** of the particle
 => particle does not always see maximum field => **effective acceleration smaller**

Transit time factor
 defined as:

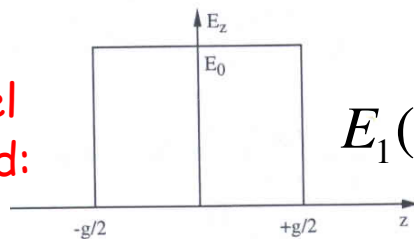
$$T_a = \frac{\text{energy gain of particle with } v = \beta c}{\text{maximum energy gain (particle with } v \rightarrow \infty)}$$

In the general case, the transit time factor is:

for $E(s, r, t) = E_1(s, r) \cdot E_2(t)$

$$T_a = \frac{\left| \int_{-\infty}^{+\infty} E_1(s, r) \cos\left(\omega_{RF} \frac{s}{v}\right) ds \right|}{\int_{-\infty}^{+\infty} E_1(s, r) ds}$$

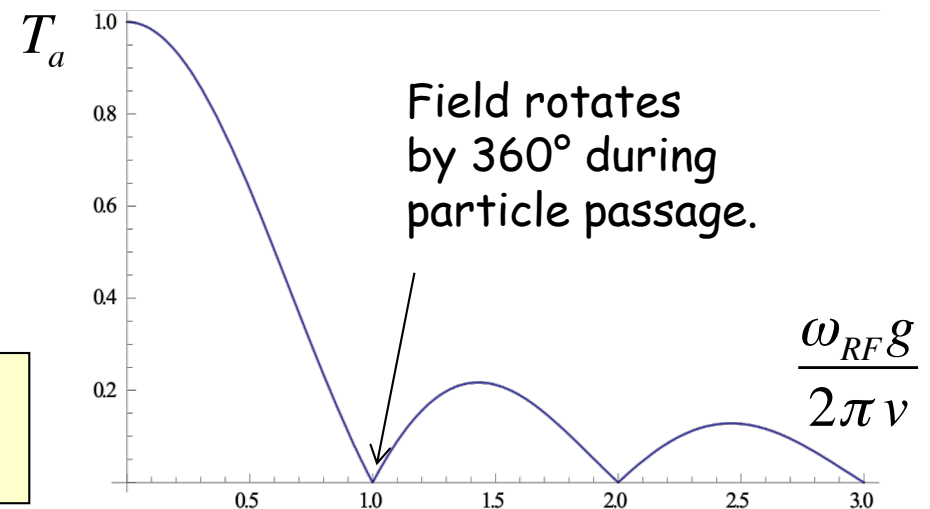
Simple model
 uniform field:



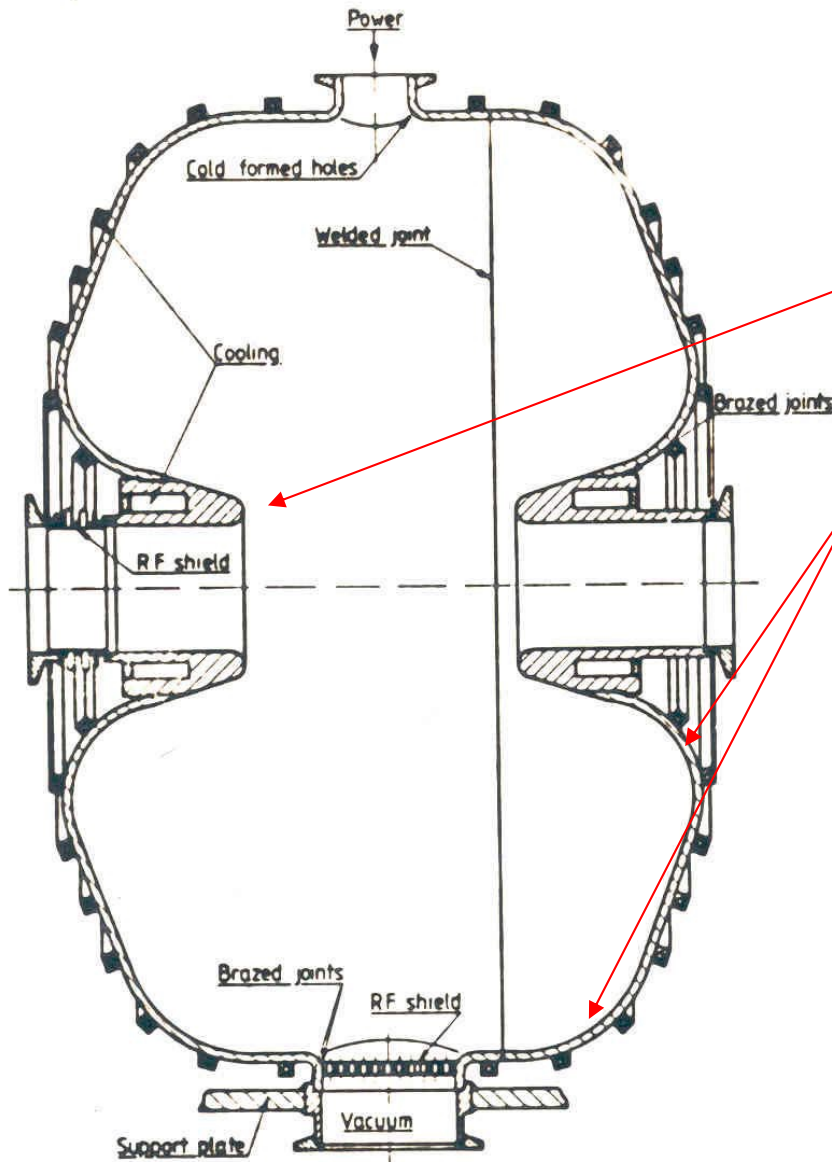
$$E_1(s, r) = \frac{V_{RF}}{g}$$

follows: $T_a = \left| \sin \frac{\omega_{RF} g}{2v} \right| / \frac{\omega_{RF} g}{2v}$

$0 < T_a < 1$, $T_a \rightarrow 1$ for $g \rightarrow 0$, smaller ω_{RF}
Important for low velocities (ions)



The Pill Box Cavity (2)



The design of a cavity can be sophisticated in order to **improve** its **performances**:

- A **nose cone** can be introduced in order to concentrate the electric field around the axis

- **Round** shaping of the **corners** allows a better distribution of the magnetic field on the surface and a reduction of the Joule losses.

It also prevents from multipactoring effects (e- emission and acceleration).

A good cavity efficiently transforms the RF power into accelerating voltage.

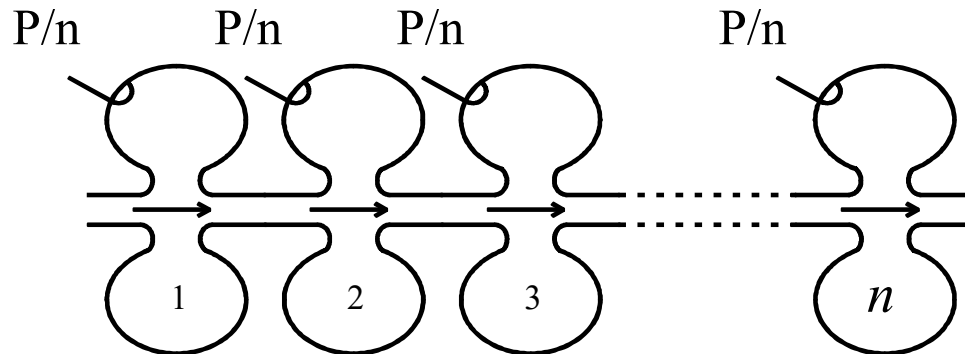
Simulation codes allow precise calculation of the properties.

Multi-Cell Cavities

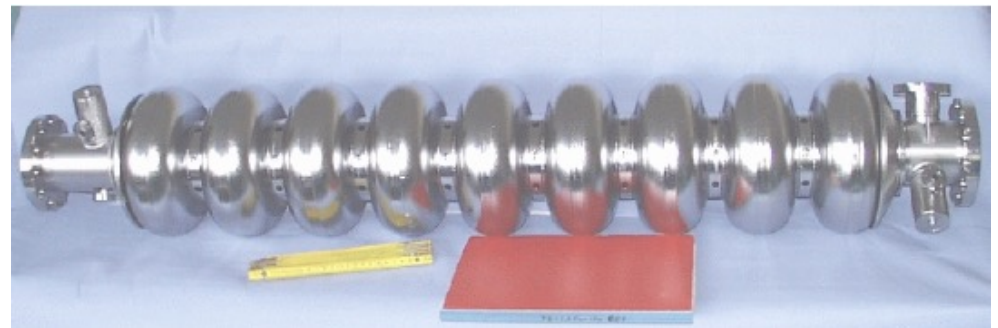
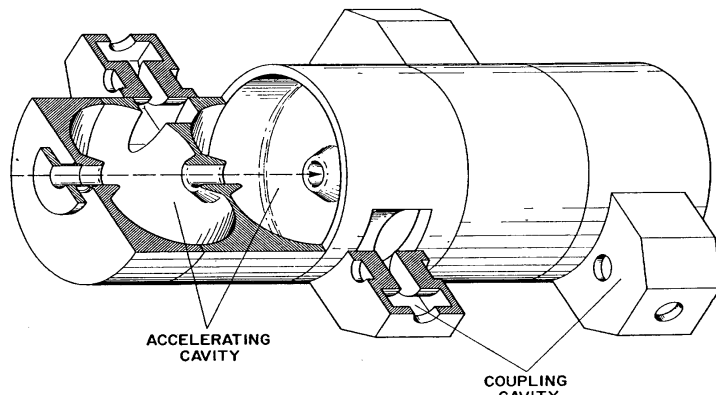
Acceleration of one cavity limited => **distribute power over several cells**

Each cavity receives P/n

Since the field is proportional \sqrt{P} , you get $\sum E_i \propto n\sqrt{P/n} = \sqrt{n}E_0$



Instead of distributing the power from the amplifier, one might as well **couple the cavities**, such that the power automatically distributes, or have a **cavity with many gaps** (e.g. drift tube linac).



Multi-Cell Cavities - Modes

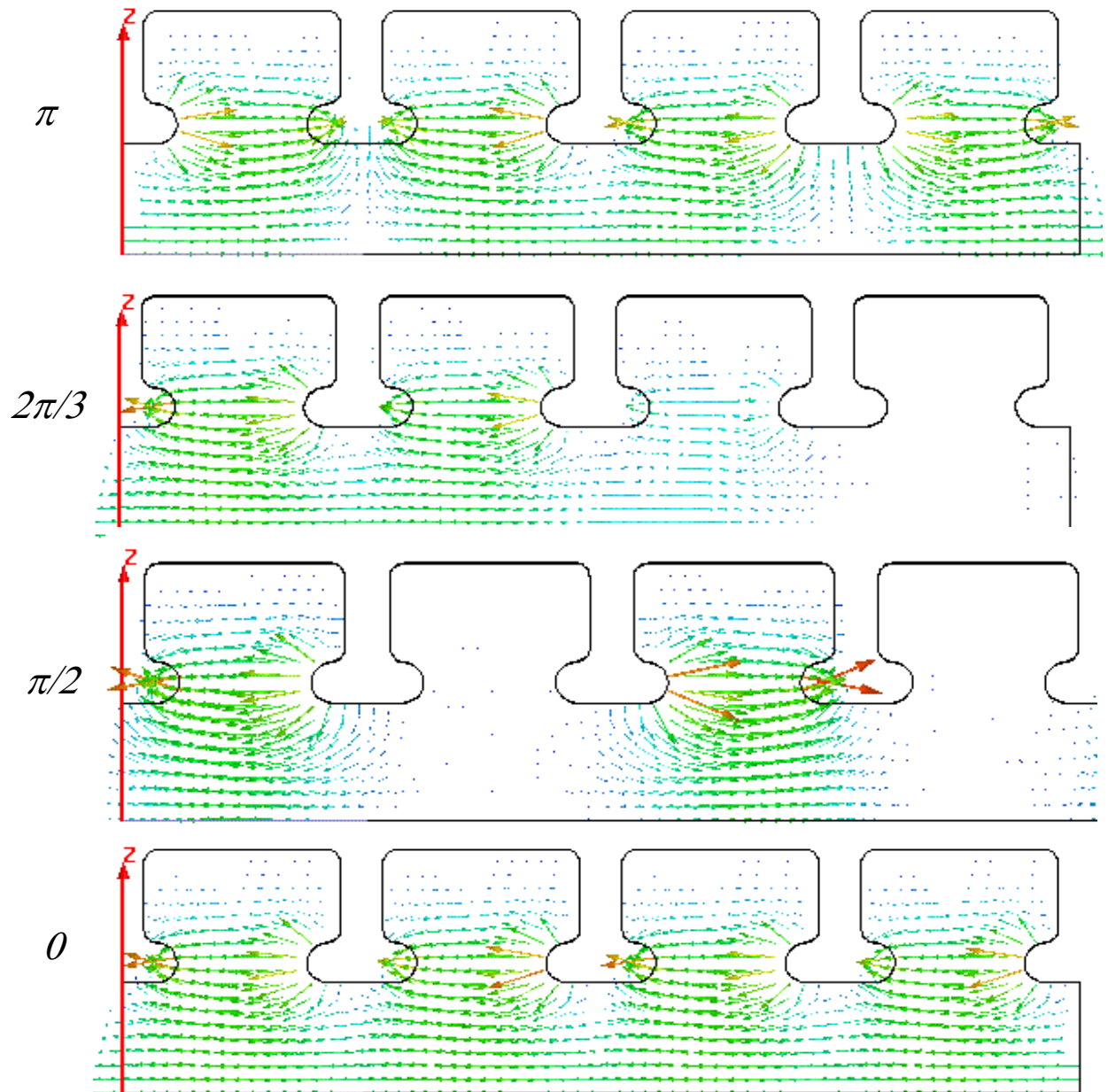
The **phase relation** between gaps is **important!**

Coupled harmonic oscillator

=> **Modes**, named after the **phase difference** between adjacent cells.

=> Different synchronism conditions for the cell length L and relativistic β

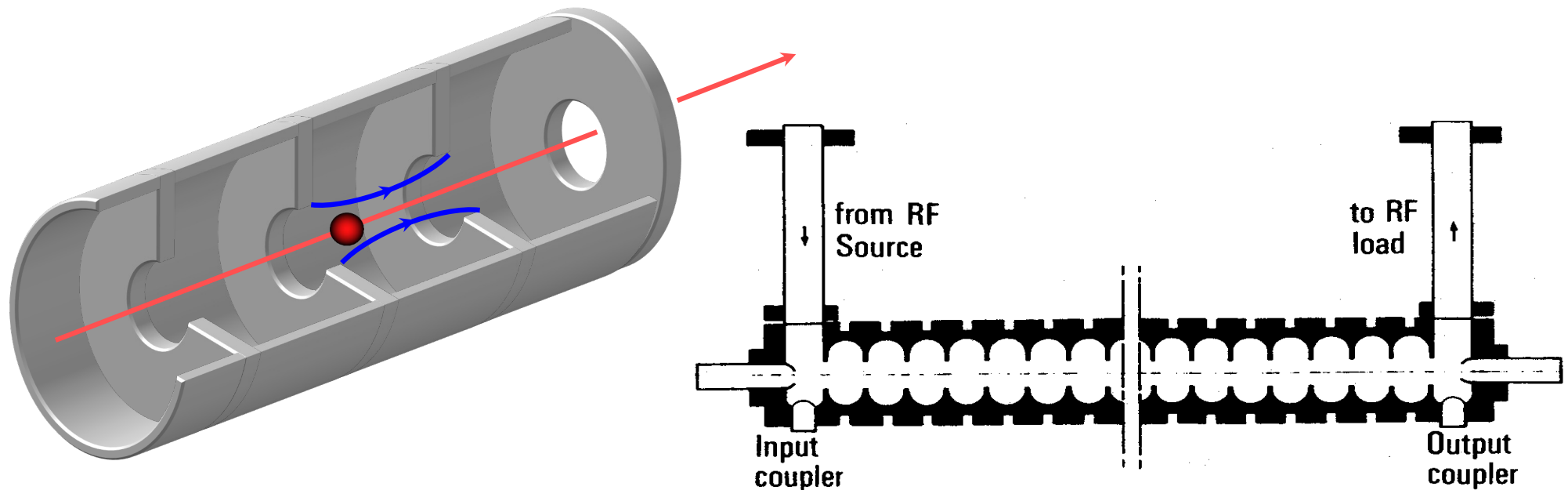
Mode	L
0 (2π)	$\beta\lambda$
$\pi/2$	$\beta\lambda/4$
$2\pi/3$	$\beta\lambda/3$
π	$\beta\lambda/2$



Disc-Loaded Traveling-Wave Structures

When particles get **ultra-relativistic** ($v \sim c$) the drift tubes become very long unless the operating frequency is increased. Late 40's the development of radar led to high power transmitters (klystrons) at very high frequencies (3 GHz).

Next came the idea of suppressing the drift tubes using **traveling waves**. However to get a continuous acceleration the phase velocity of the wave needs to be adjusted to the particle velocity. In waveguide: $v_{ph} > c$



solution: slow wave guide with irises ==> iris loaded structure

Cavity Parameters: Quality Factor Q

The **total energy stored** is $W = \iiint_{cavity} \left(\frac{\epsilon}{2} |\vec{E}|^2 + \frac{\mu}{2} |\vec{H}|^2 \right) dV.$

- **Quality Factor Q** (caused by wall losses) defined as

$$Q_0 = \frac{\omega_0 W}{P_{loss}}$$

Ratio of stored energy W
and dissipated power P_{loss}
on the walls in one RF cycle

The Q factor determines the maximum energy the cavity can fill to with a given input power.

Larger Q => less power needed to sustain stored energy.

The Q factor is 2π times the number of rf cycles it takes to dissipate the energy stored in the cavity (down by $1/e$).

- function of the geometry and the **surface resistance of the material**:
superconducting (niobium) : $Q = 10^{10}$
normal conducting (copper) : $Q = 10^4$

Important Parameters of Accelerating Cavities

- Accelerating voltage V_{acc}

$$V_{acc} = \int_{-\infty}^{\infty} E_z e^{-i\frac{\omega z}{\beta c}} dz$$

Measure of the acceleration

- R upon Q

$$\frac{R}{Q} = \frac{|V_{acc}|^2}{2\omega_0 W}$$

Relationship between acceleration V_{acc} and stored energy W

independent from material!

Attention: Different definitions are used!

- Shunt Impedance R

$$R = \frac{|V_{acc}|^2}{2P_{loss}}$$

Relationship between acceleration V_{acc} and wall losses P_{loss}

depends on
- material
- cavity mode
- geometry

Important Parameters of Accelerating Cavities (cont.)

- Fill Time t_F

- standing wave cavities:

$$P_{loss} = -\frac{dW}{dt} = \frac{\omega_0}{Q} W$$

Exponential decay of the stored energy W due to losses

$$t_F = \frac{2Q}{\omega_0}$$

time for the field to decrease by $1/e$ after the cavity has been filled
measure of how fast the stored energy is dissipated on the wall

Several fill times needed to fill the cavity!

- travelling wave cavities:

time needed for the electromagnetic energy to fill the cavity of length L

$$t_F = \int_0^L \frac{dz}{v_g(z)}$$

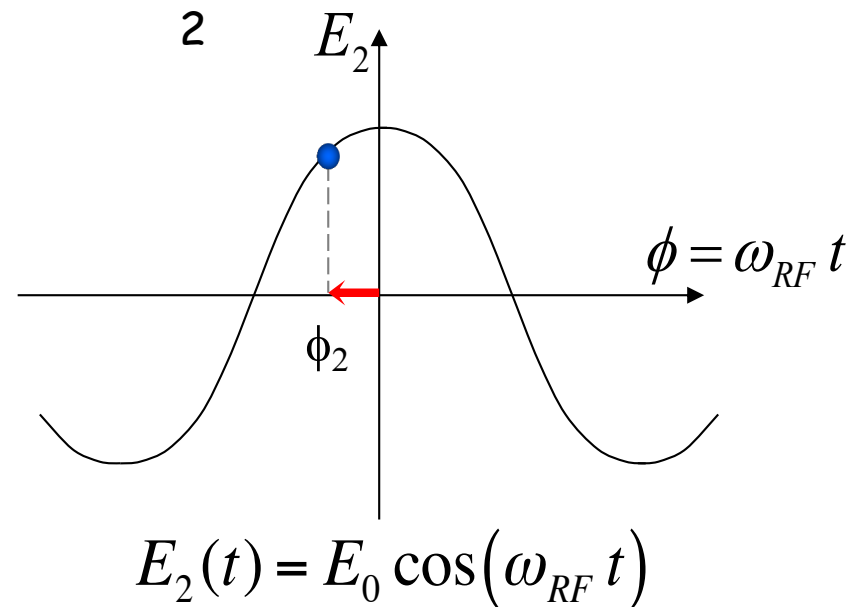
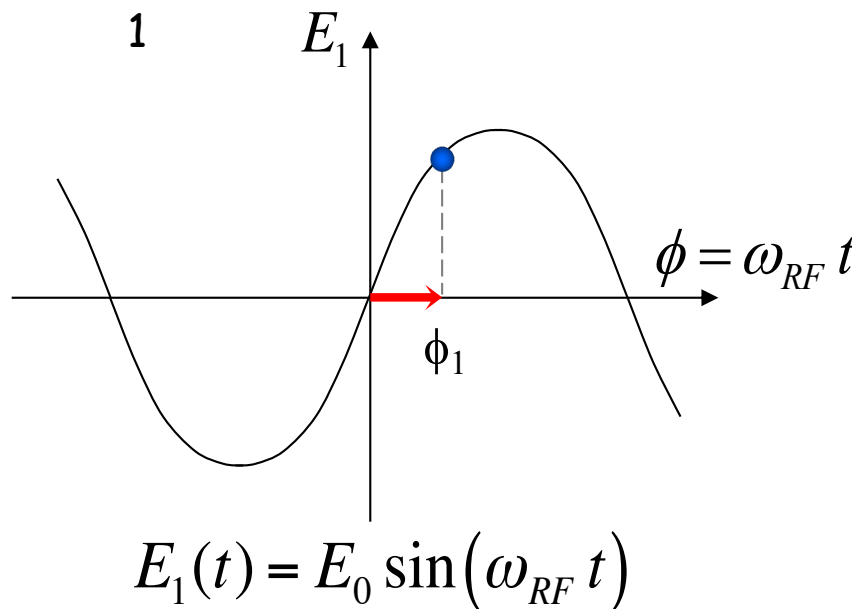
v_g : velocity at which the energy propagates through the cavity

Cavity is completely filled after 1 fill time!

Common Phase Conventions

1. For **circular accelerators**, the origin of time is taken at the **zero crossing** of the RF voltage with positive slope
2. For **linear accelerators**, the origin of time is taken at the positive **crest** of the RF voltage

Time $t = 0$ chosen such that:

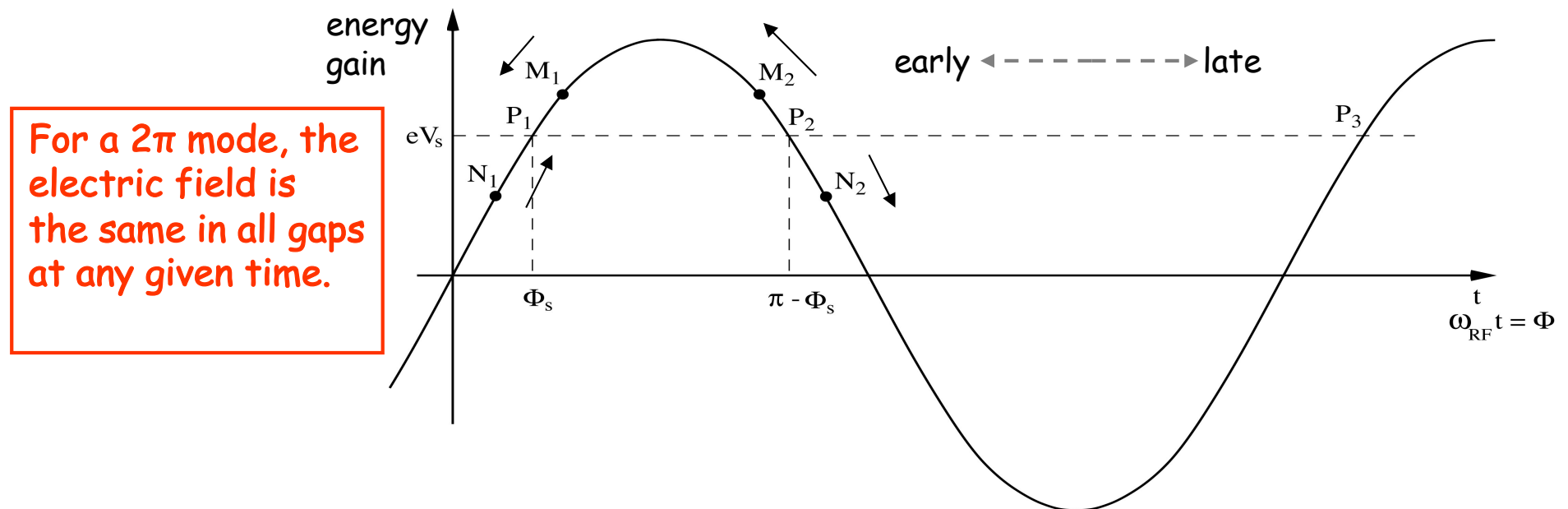


3. I will stick to **convention 1** in the following to avoid confusion

Principle of Phase Stability (Linac)

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

$eV_s = e\hat{V} \sin \Phi_s$ is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1, P_2, \dots are fixed points.



If an **energy increase** is transferred into a **velocity increase** \Rightarrow

M_1 & N_1 will move towards P_1 \Rightarrow **stable**

M_2 & N_2 will go away from P_2 \Rightarrow **unstable**

(Highly relativistic particles have no significant velocity change)

Energy-phase Oscillations (Small Amplitude) (1)

- Rate of **energy gain** for the **synchronous particle**: $\frac{dE_s}{dz} = \frac{dp_s}{dt} = eE_0 \sin \phi_s$

- Use **variables relative to synchronous particle**:

$$w = W - W_s = E - E_s$$

$$\varphi = \phi - \phi_s$$

- Rate of **energy gain** for a **non-synchronous particle**:

$$\frac{dw}{dz} = eE_0 [\sin(\phi_s + \varphi) - \sin \phi_s] \approx eE_0 \cos \phi_s \cdot \varphi \quad (\text{small } \varphi)$$

- Rate of change of the **phase** with respect to the synchronous one:

$$\frac{d\varphi}{dz} = \omega_{RF} \left(\frac{dt}{dz} - \left(\frac{dt}{dz} \right)_s \right) = \omega_{RF} \left(\frac{1}{v} - \frac{1}{v_s} \right) \approx -\frac{\omega_{RF}}{v_s^2} (v - v_s)$$

Leads finally to:

$$\frac{d\varphi}{dz} = -\frac{\omega_{RF}}{m_0 v_s^3 \gamma_s^3} w$$

Energy-phase Oscillations (Small Amplitude) (2)

Combining the two 1st order equations into a 2nd order equation gives the equation of a **harmonic oscillator**:

$$\frac{d^2\phi}{dz^2} + \Omega_s^2\phi = 0 \quad \text{with}$$

$$\Omega_s^2 = \frac{eE_0\omega_{RF}\cos\phi_s}{m_0\gamma_s^3 v_s^3}$$

Slower for higher energy!

Stable harmonic oscillations imply:

$$\Omega_s^2 > 0 \quad \text{and real}$$

hence: $\cos\phi_s > 0$

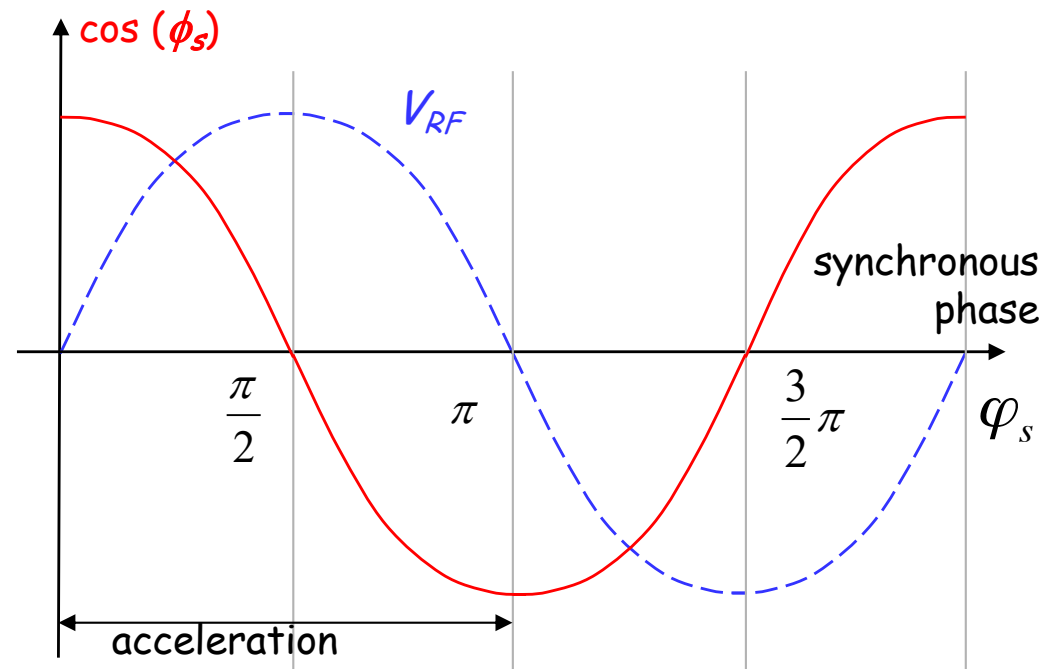
And since acceleration also means:

$$\sin\phi_s > 0$$

You finally get the result for the **stable phase range**:

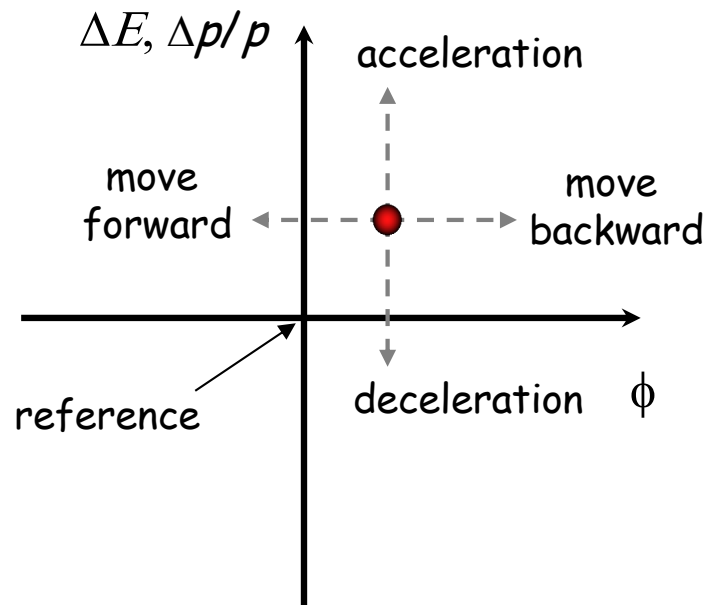
$$0 < \phi_s < \frac{\pi}{2}$$

Positive rising RF slope!

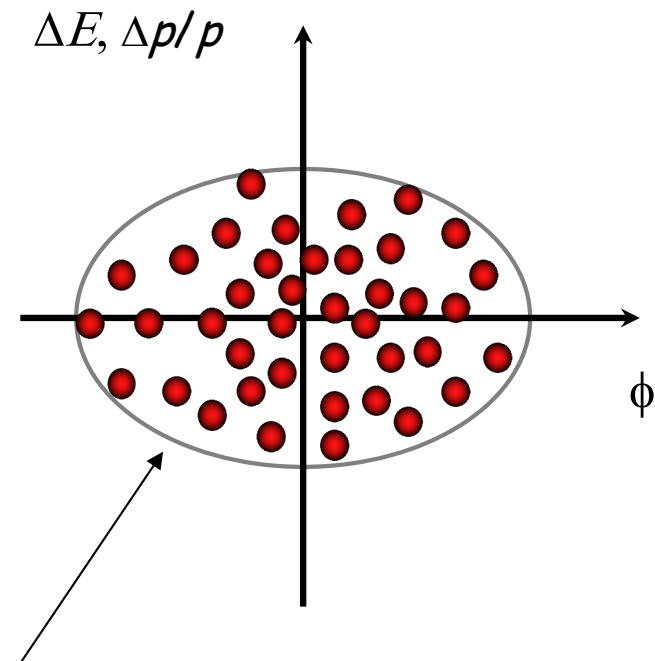


Longitudinal phase space

The **energy - phase oscillations** can be drawn in **phase space**:



The particle trajectory in the phase space ($\Delta p/p, \phi$) describes its longitudinal motion.



Emittance: phase space area including all the particles

NB: if the emittance contour correspond to a possible orbit in phase space, its shape does not change with time (matched beam)

Summary up to here...

- Acceleration by electric fields, static fields limited
=> time-varying fields, **that's why we use RF**
- **Synchronous condition** needs to be fulfilled for acceleration
- Accelerator designed for a **synchronous particle** with **synchronous phase**
- Particles perform **oscillation** around synchronous phase
- Stable acceleration on the rising RF slope in a linac.
- visualize oscillations in phase space
- Electrons are quickly relativistic, speed does not change
- Protons and ions
 - significantly change velocity at lower energy
 - need changing structure geometry / tunable structures

Summary: Relativity + Energy Gain

Newton-Lorentz Force $\vec{F} = \frac{d\vec{p}}{dt} = e \left(\vec{E} + \vec{v} \times \vec{B} \right)$ 2nd term always perpendicular to motion => no acceleration

Relativistics Dynamics

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \quad \gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \beta^2}}$$

$$p = mv = \frac{E}{c^2} \beta c = \beta \frac{E}{c} = \beta \gamma m_0 c$$

$$E^2 = E_0^2 + p^2 c^2 \quad \longrightarrow \quad dE = v dp$$

$$\frac{dE}{dz} = v \frac{dp}{dz} = \frac{dp}{dt} = e E_z$$

$$dE = dW = e E_z dz \quad \rightarrow \quad W = e \int E_z dz$$

RF Acceleration

$$E_z = \hat{E}_z \sin \omega_{RF} t = \hat{E}_z \sin \phi(t)$$

$$\int \hat{E}_z dz = \hat{V}$$

$$W = e \hat{V} \sin \phi$$

(neglecting transit time factor)

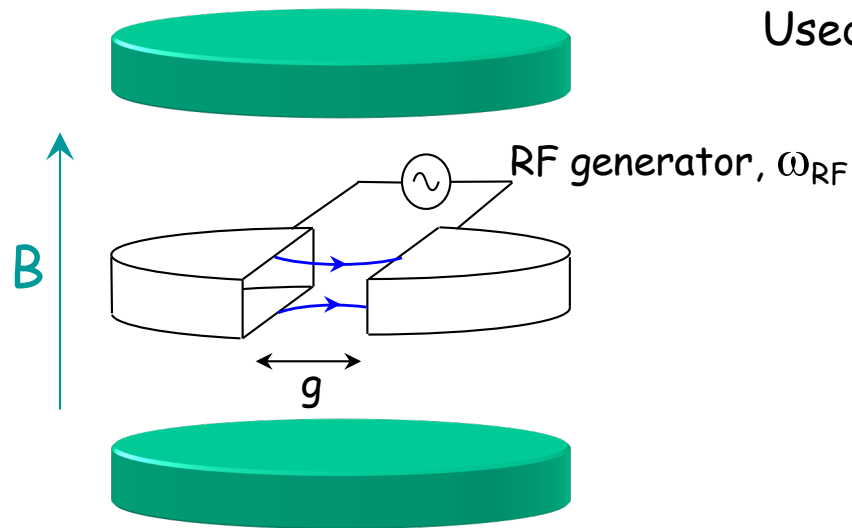
The field will change during the passage of the particle through the cavity

=> effective energy gain is lower

Circular accelerators

Cyclotron
Synchrotron

Circular accelerators: Cyclotron



$B = \text{constant}$

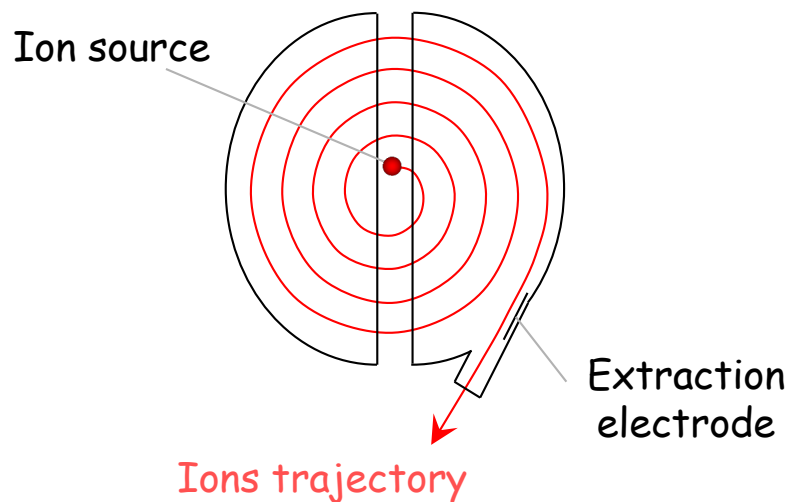
$\omega_{RF} = \text{constant}$

Synchronism condition



$$\omega_s = \omega_{RF}$$

$$2\pi \rho = v_s T_{RF}$$



Cyclotron frequency

$$\omega = \frac{q B}{m_0 \gamma}$$

1. γ increases with the energy
 \Rightarrow no exact synchronism
2. if $v \ll c \Rightarrow \gamma \cong 1$

Animation: <https://phyanim.sciences.univ-nantes.fr/Meca/Charges/cyclotron.php>

Circular accelerators: Cyclotron



Courtesy Berkeley Lab,
<https://www.youtube.com/watch?v=cutKuFxeXmQ>

IAS Superconductivity, Saskatoon, Canada, July 2023

Cyclotron / Synchrocyclotron



TRIUMF 520 MeV cyclotron

Vancouver - Canada



CERN 600 MeV synchrocyclotron

Synchrocyclotron: Same as cyclotron, except a modulation of ω_{RF}

B = constant

$\gamma \omega_{RF}$ = constant

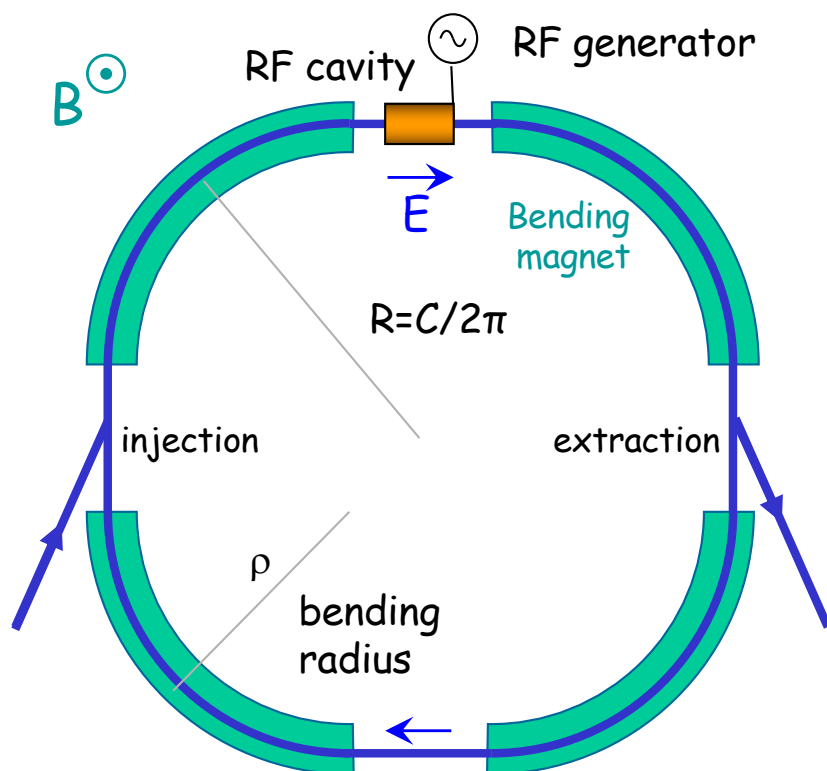
ω_{RF} decreases with time

The condition:

$$\omega_s(t) = \omega_{RF}(t) = \frac{q B}{m_0 \gamma(t)}$$

Allows to go beyond the non-relativistic energies

Circular accelerators: The Synchrotron



Synchronism condition



1. Constant orbit during acceleration
2. To keep particles on the closed orbit, B should increase with time
3. ω and ω_{RF} increase with energy

RF frequency can be multiple of revolution frequency

$$\omega_{RF} = h\omega$$

$$T_s = h T_{RF}$$

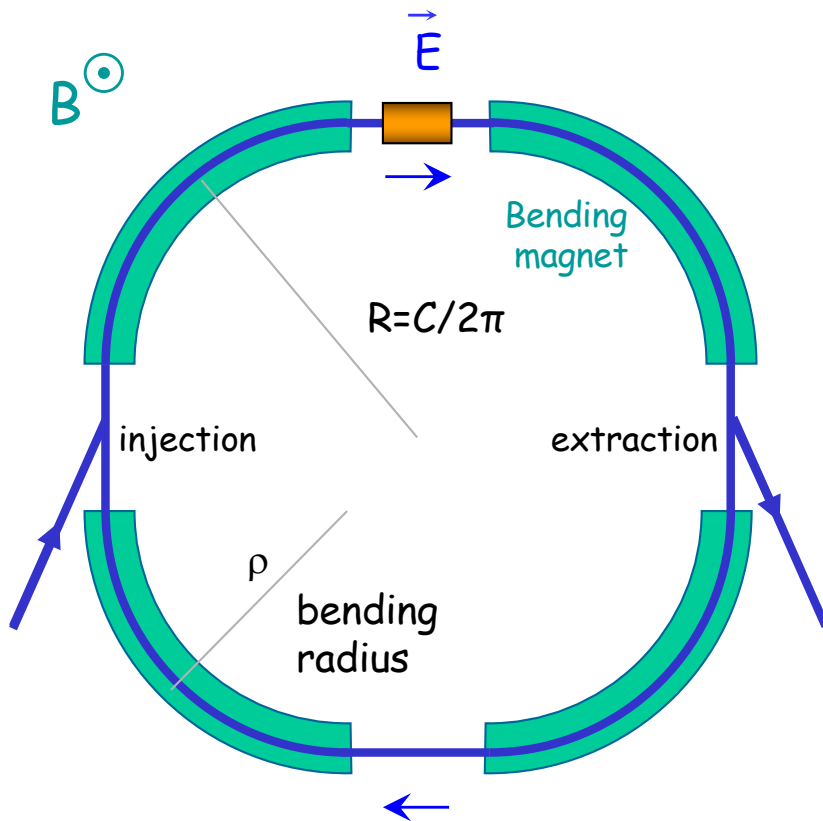
$$\frac{2\pi R}{v_s} = h T_{RF}$$

h integer,
harmonic number:
 number of RF cycles
 per revolution

h is the **maximum number of bunches** in the synchrotron.
 Normally less bunches due to gaps for kickers, collision constraints,...

The Synchrotron

The **synchrotron** is a synchronous accelerator since there is a **synchronous RF phase** for which the energy gain **fits** the **increase of the magnetic field** at each turn. That implies the following operating conditions:



$$eV \sin \phi \longrightarrow \text{Energy gain per turn}$$

$$\phi = \phi_s = cte \longrightarrow \text{Synchronous particle}$$

$$\omega_{RF} = h\omega \longrightarrow \text{RF synchronism (h - harmonic number)}$$

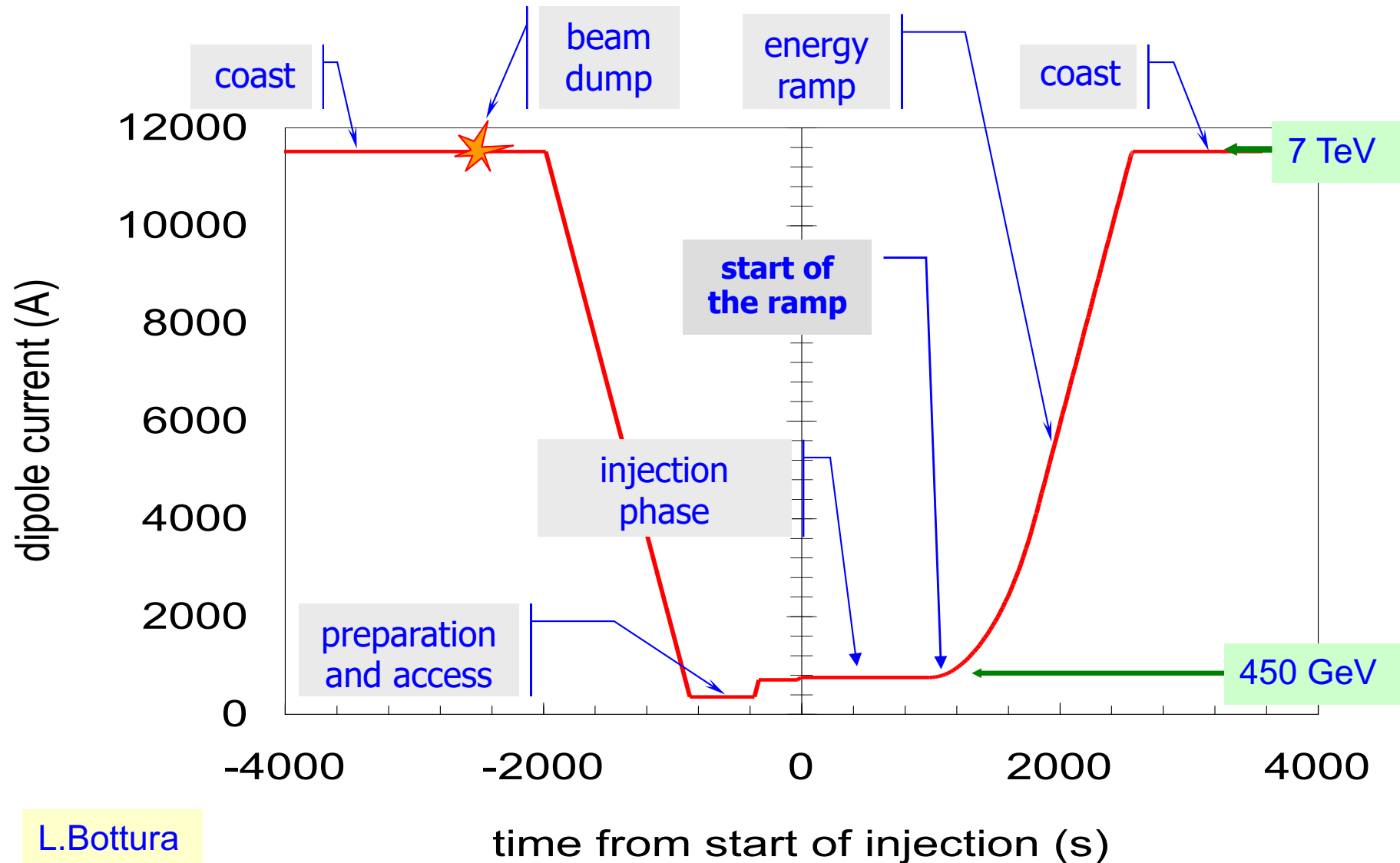
$$\rho = cte \quad R = cte \longrightarrow \text{Constant orbit}$$

$$B\rho = P/e \Rightarrow B \longrightarrow \text{Variable magnetic field}$$

If $v \approx c$, ω hence ω_{RF} remain constant (ultra-relativistic e^-)

The Synchrotron - LHC Operation Cycle

The magnetic **field** (dipole current) is **increased during** the **acceleration**.



L.Bottura

The Synchrotron - Energy ramping

Energy ramping by increasing the B field (frequency has to follow v):

$$p = eB\rho \quad \rho \text{ const.} \Rightarrow \frac{dp}{dt} = e\rho \dot{B} \Rightarrow (\Delta p)_{turn} = e\rho \dot{B} T_r = \frac{2\pi e\rho R \dot{B}}{v}$$

With

$$E^2 = E_0^2 + p^2 c^2 \Rightarrow \Delta E = v \Delta p \quad (\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho R \dot{B} = e\hat{V} \sin \phi_s$$

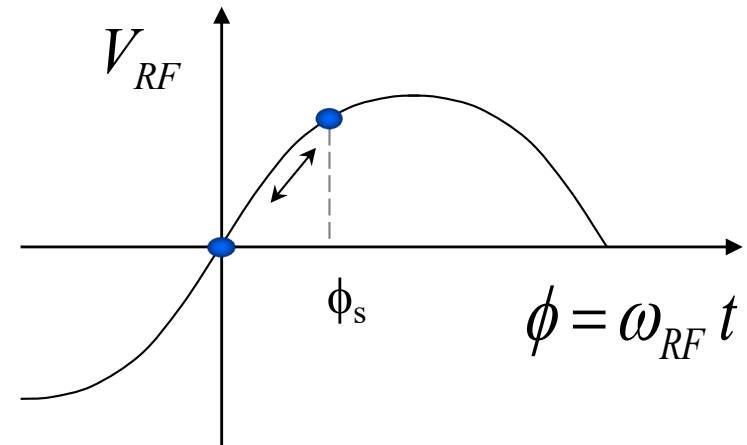
Synchronous phase ϕ_s changes during energy ramping

$$\sin \phi_s = 2\pi \rho R \frac{\dot{B}}{\hat{V}_{RF}}$$



$$\phi_s = \arcsin\left(2\pi \rho R \frac{\dot{B}}{\hat{V}_{RF}}\right)$$

- The **synchronous phase** depends on
 - the **change of the magnetic field**
 - and the RF voltage



The Synchrotron - Frequency change

During the energy ramping, **the RF frequency increases** to follow the increase of the revolution frequency :

$$\omega = \frac{\omega_{RF}}{h} = \omega(B, R_s)$$

Hence:
$$\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi} \frac{ec^2}{E_s(t)} \frac{\rho}{R_s} B(t) \quad \left(\text{using } p(t) = eB(t)\rho, \quad E = mc^2 \right)$$

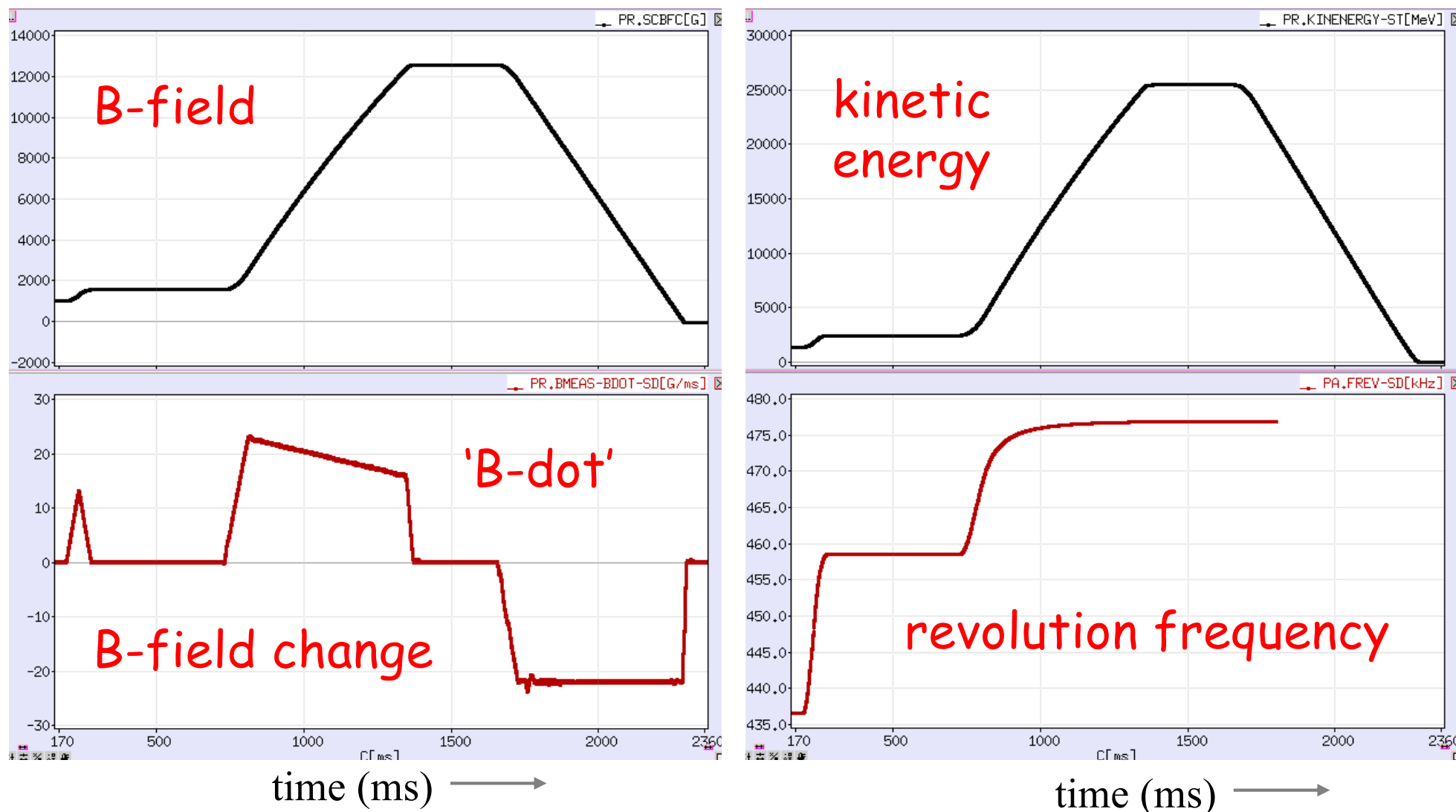
Since $E^2 = (m_0c^2)^2 + p^2c^2$ **the RF frequency must follow the variation of the B field with the law**

$$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s} \left\{ \frac{B(t)^2}{(m_0c^2 / ec\rho)^2 + B(t)^2} \right\}^{1/2}$$

RF frequency program during acceleration determined by B-field

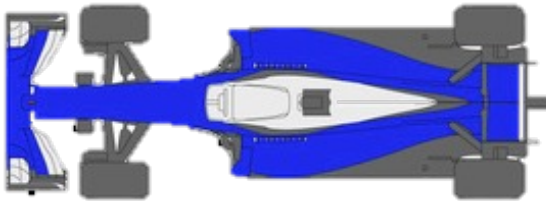
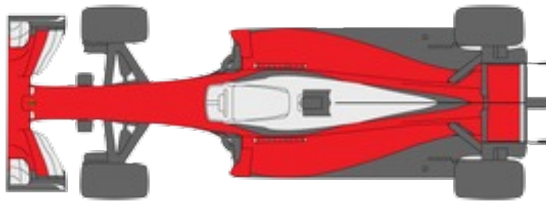
Example: PS - Field / Frequency change

During the energy ramping, the **B-field** and the **revolution frequency** increase



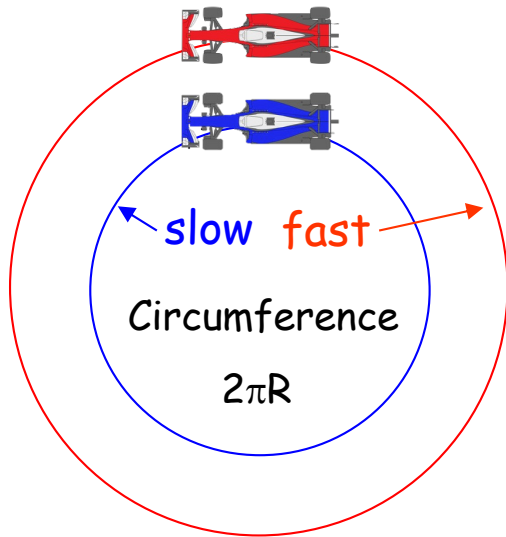
Overtaking in a Formula 1 Race





Overtaking in a Formula 1 Race

Overtaking in a Formula 1 Race



v =speed of the car

R =track physical radius

T =revolution period

f_r =revolution frequency

A F1 car wants to overtake another car! It will have a

- a **different track length** due to a 'dispersion orbit'
- and a **different velocity**.

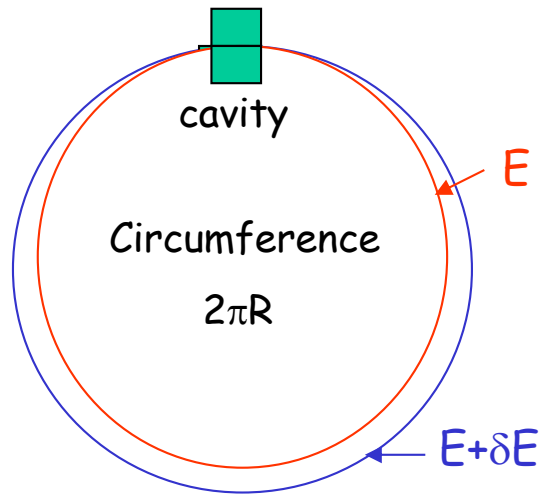
$$T = \frac{L}{v} = \frac{2\pi R}{v} \quad \text{and} \quad f_r = \frac{1}{T} = \frac{v}{2\pi R}$$

$$\Rightarrow \frac{\Delta T}{T} = \frac{\Delta R}{R} - \frac{\Delta v}{v}$$

The winner depends on the **relative change in speed** compared to the **relative change in track length**!

If the **relative change in speed** is larger than the **relative change in track length** \Rightarrow the **red** car will win!

Overtaking in a Synchrotron



A particle with a momentum deviation will have a

- dispersion orbit and a **different orbit length**
- a **different velocity**.

As a result of both effects the **revolution time changes** with a "**slip factor η** ":

$$\eta = \frac{dT/T}{dp/p}$$

Note: you also find η defined with a minus sign!

p =particle momentum

R =synchrotron physical radius

f_r =revolution frequency

Effect from orbit defined by **Momentum compaction factor:**

$$\alpha_c = \frac{dL/L}{dp/p}$$

Property of the **beam optics**:
(derivation see appendix)

$$\alpha_c = \frac{1}{L} \int_C \frac{D_x(s)}{\rho(s)} ds_0$$

Dispersion Effects - Revolution Period

The **two effects** of the **orbit length** and the particle **velocity** change the revolution period as:

$$T = \frac{L}{\beta c} \quad \Rightarrow \quad \frac{dT}{T} = \frac{dL}{L} - \frac{d\beta}{\beta} = \alpha_c \frac{dp}{p} - \frac{d\beta}{\beta}$$

↑
definition of momentum
compaction factor

$$\frac{dT}{T} = \left(\alpha_c - \frac{1}{\gamma^2} \right) \frac{dp}{p}$$

$$p = mv = \beta\gamma \frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d(1-\beta^2)^{-1/2}}{(1-\beta^2)^{-1/2}} = \underbrace{(1-\beta^2)^{-1}}_{\gamma^2} \frac{d\beta}{\beta}$$

**Slip
factor:**

$$\eta = \alpha_c - \frac{1}{\gamma^2}$$

or

$$\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2}$$

with

$$\gamma_t = \frac{1}{\sqrt{\alpha_c}}$$

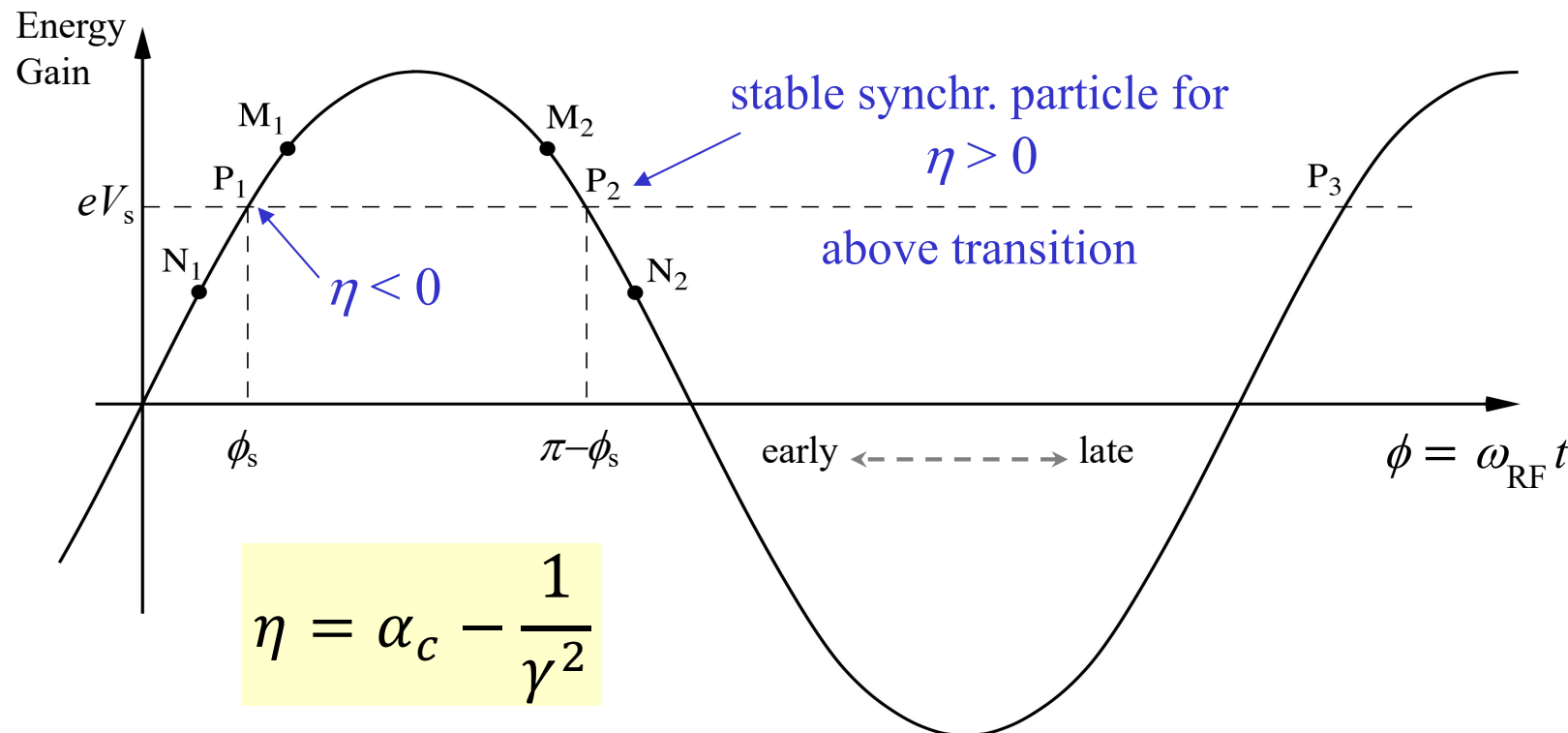
Note: you also find η defined with a minus sign!

At **transition energy**, $\eta = 0$, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Phase Stability in a Synchrotron

From the definition of η it is clear that an **increase in momentum** gives

- **below transition** ($\eta < 0$) a **higher revolution frequency** (increase in velocity dominates) while
- **above transition** ($\eta > 0$) a **lower revolution frequency** ($v \approx c$ and longer path) where the momentum compaction (generally > 0) dominates.

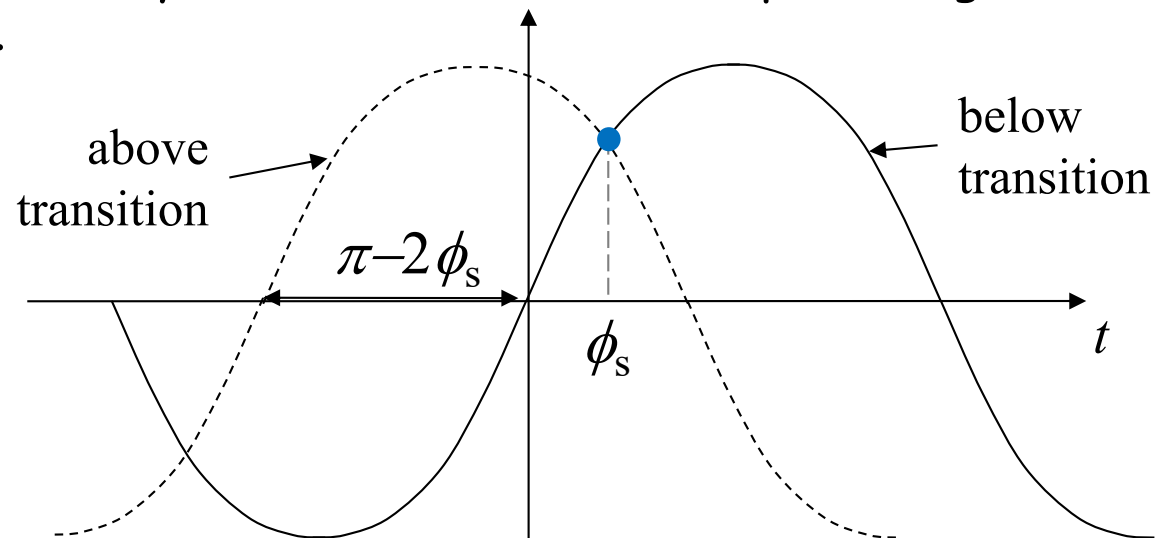


Crossing Transition

At **transition**, the velocity change and the path length change with momentum compensate each other. So the revolution frequency there is independent from the momentum deviation.

Crossing transition during acceleration makes the previous stable synchronous phase unstable. The RF system needs to make a rapid change of the RF phase, a '**phase jump**'.

$$\alpha_c \sim \frac{1}{Q_x^2} \quad \gamma_t = \frac{1}{\sqrt{\alpha_c}} \sim Q_x$$



In the PS: γ_t is at $\sim 6 \text{ GeV}$

In the SPS: $\gamma_t = 22.8$, injection at $\gamma = 27.7$

=> no transition crossing!

In the LHC: γ_t is at $\sim 55 \text{ GeV}$, also far below injection energy

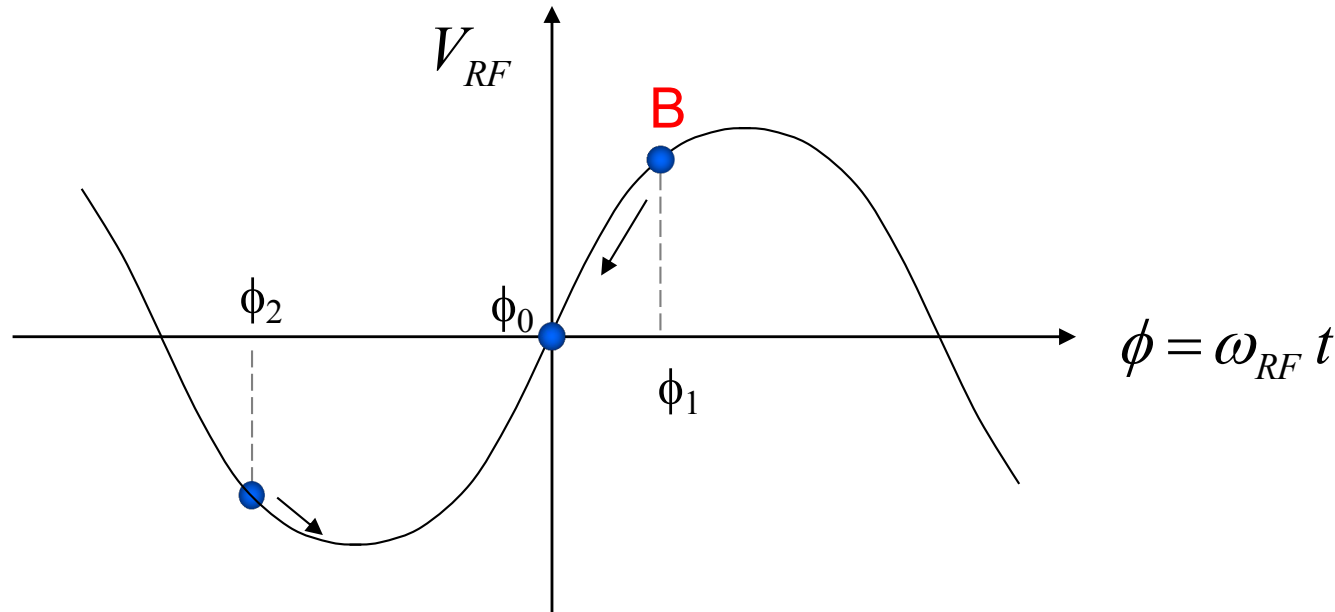
Transition crossing not needed in leptons machines, why?

Dynamics: Synchrotron oscillations

Simple case (no accel.): $B = \text{const.}$, below transition $\gamma < \gamma_{tr}$

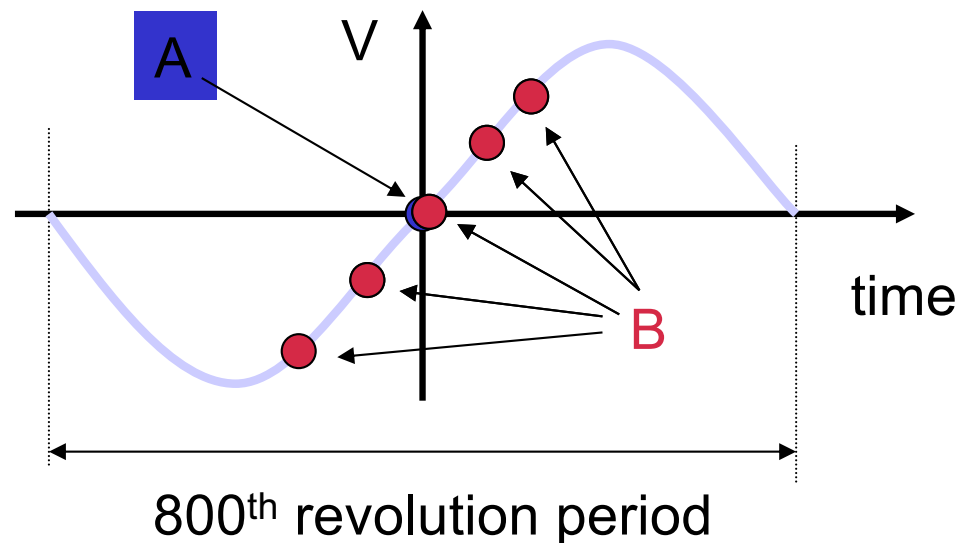
The phase of the synchronous particle must therefore be $\phi_0 = 0$.

- ϕ_1
- The particle **B** is accelerated
 - Below transition, an increase in energy means an increase in revolution frequency
 - The particle arrives earlier - tends toward ϕ_0



- ϕ_2
- The particle is decelerated
 - decrease in energy - decrease in revolution frequency
 - The particle arrives later - tends toward ϕ_0

Synchrotron oscillations



Particle **B** performs **Synchrotron Oscillations** around synchronous particle **A**.

The amplitude depends on the initial phase and energy.

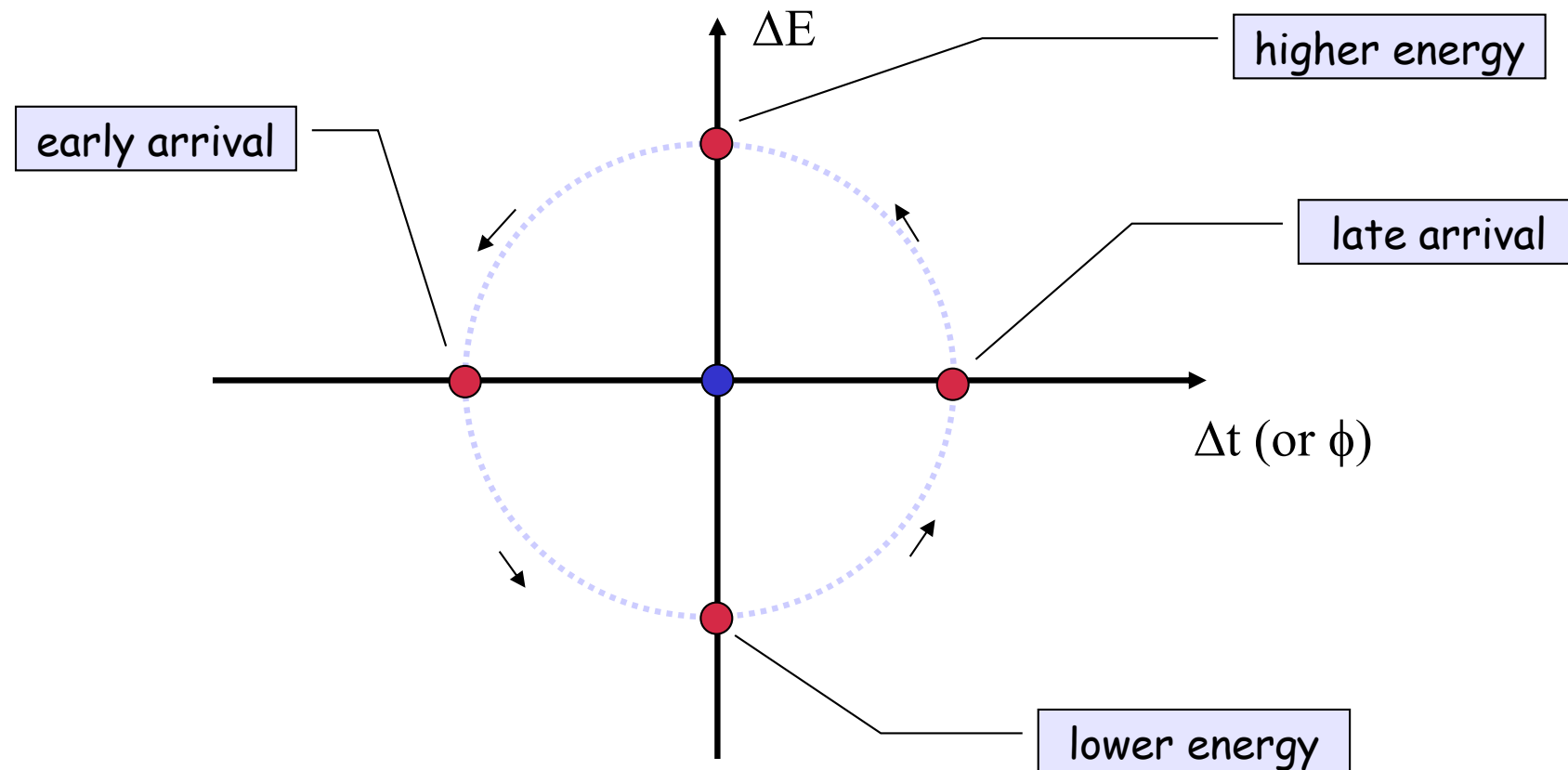
The **oscillation frequency** is much **slower than** in the **transverse** plane. It takes a large number of revolutions for one complete oscillation. The restoring electric force is smaller than the magnetic force.

- proton synchrotrons of the order of 1000 turns
- electron storage rings of the order of ~10 turns

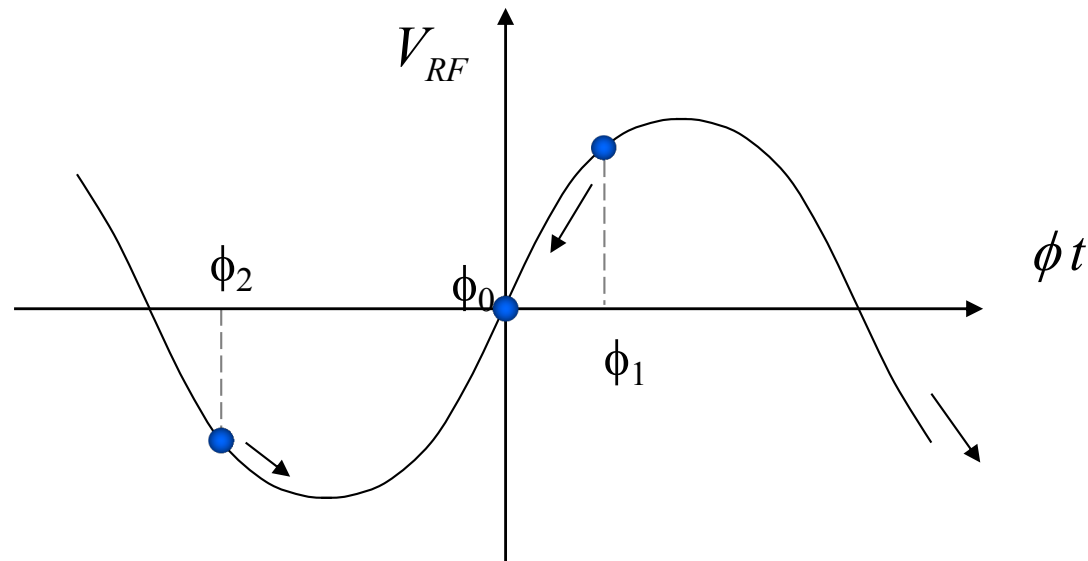
Longitudinal Phase Space Motion

Particle **B** performs a **synchrotron oscillation** around the synchronous particle **A**

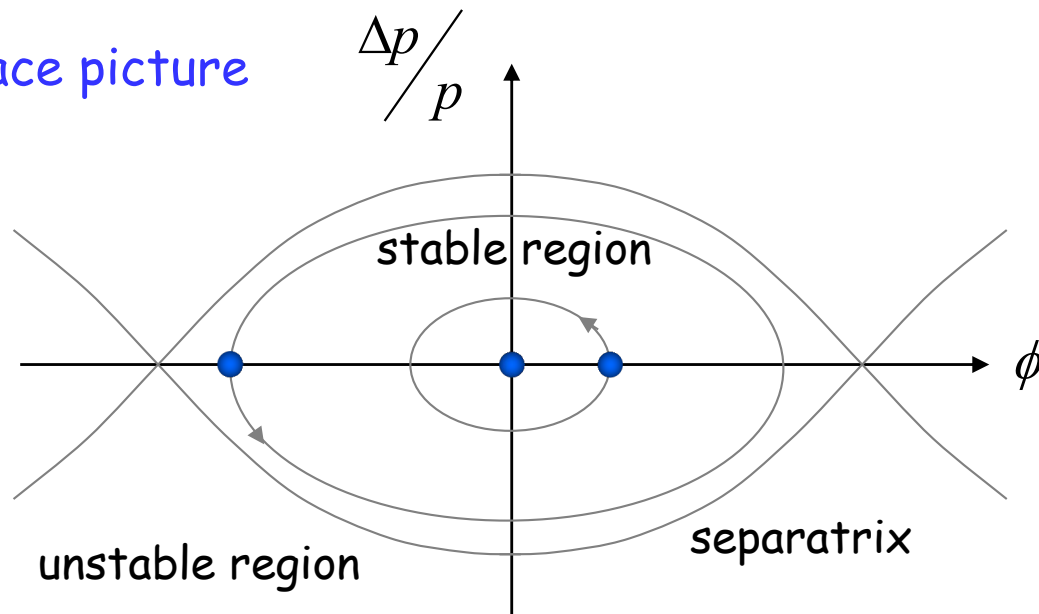
Plotting this motion in longitudinal phase space gives:



Synchrotron oscillations - No acceleration

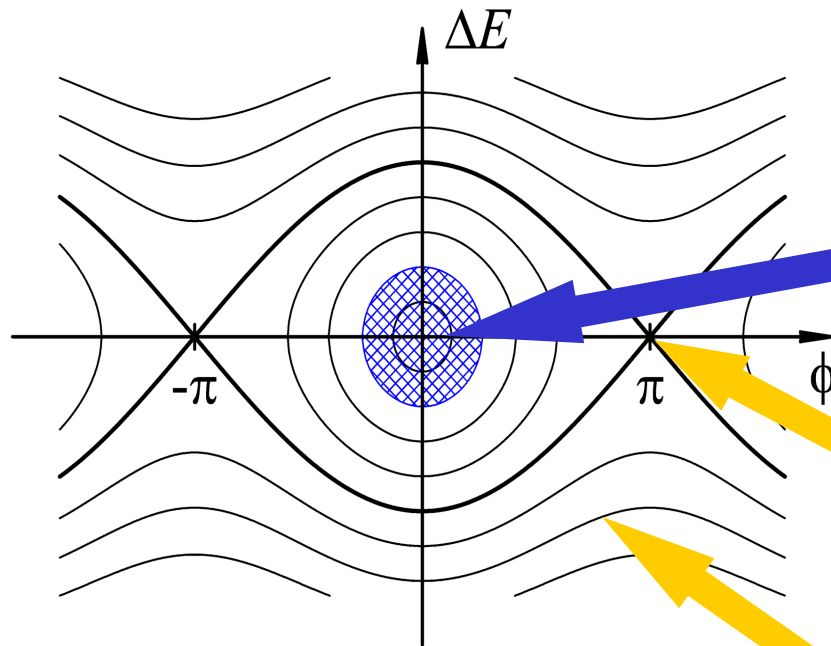


Phase space picture



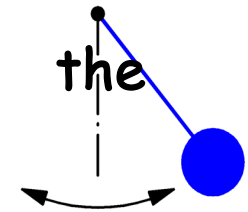
Synchrotron motion in phase space

ΔE - ϕ phase space of a **stationary bucket**
(when there is **no acceleration**)



Dynamics of a particle
Non-linear, conservative
oscillator \rightarrow e.g. pendulum

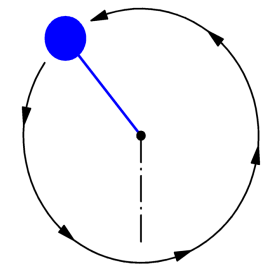
Particle inside
the separatrix:



Particle at the
unstable fix-point



Particle outside
the separatrix:

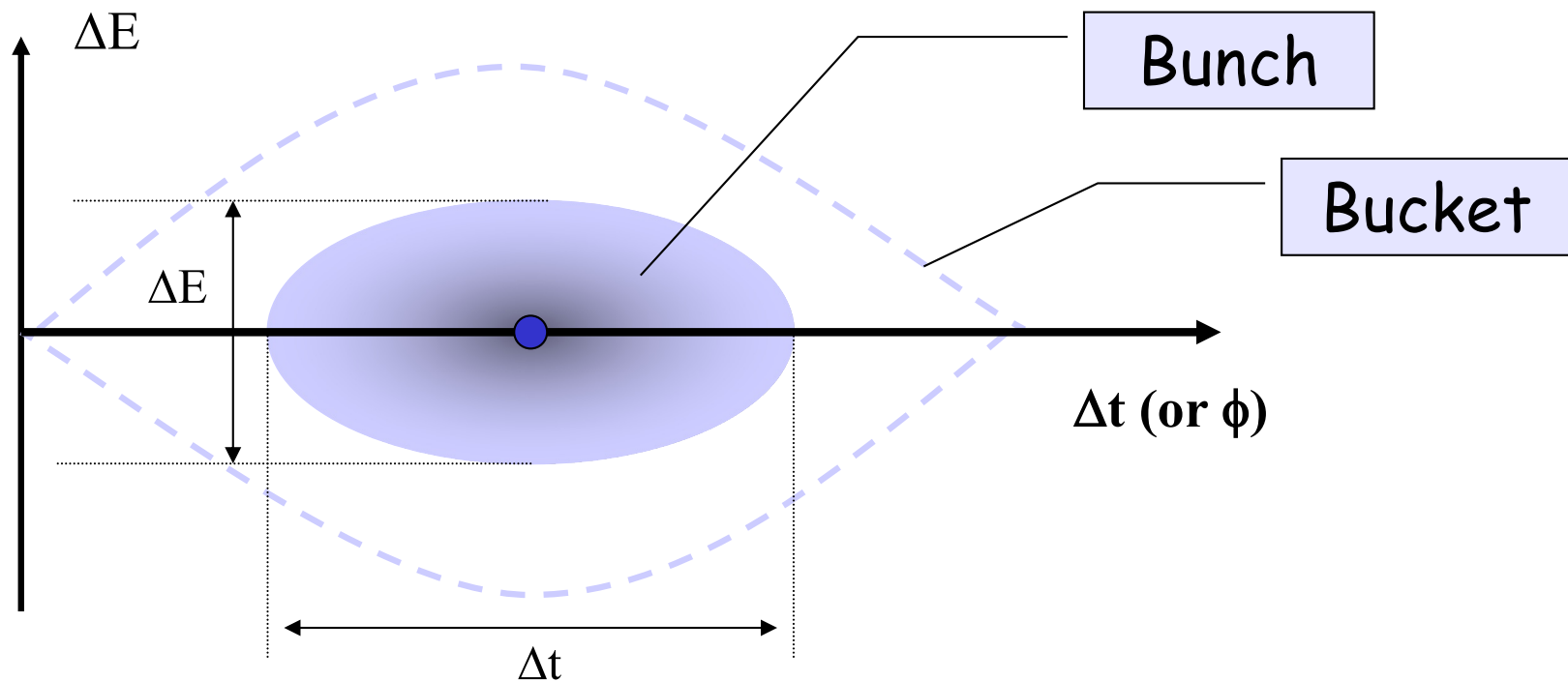


Bucket area:
area enclosed by the separatrix
 \Rightarrow **longitudinal Acceptance** [eVs]

The area covered by particles is the
longitudinal emittance.

(Stationary) Bunch & Bucket

The **bunches** of the beam **fill** usually **a part of** the **bucket** area.



Bucket area = longitudinal Acceptance [eVs]

Bunch area = longitudinal beam emittance = $4\pi \sigma_E \sigma_t$ [eVs]

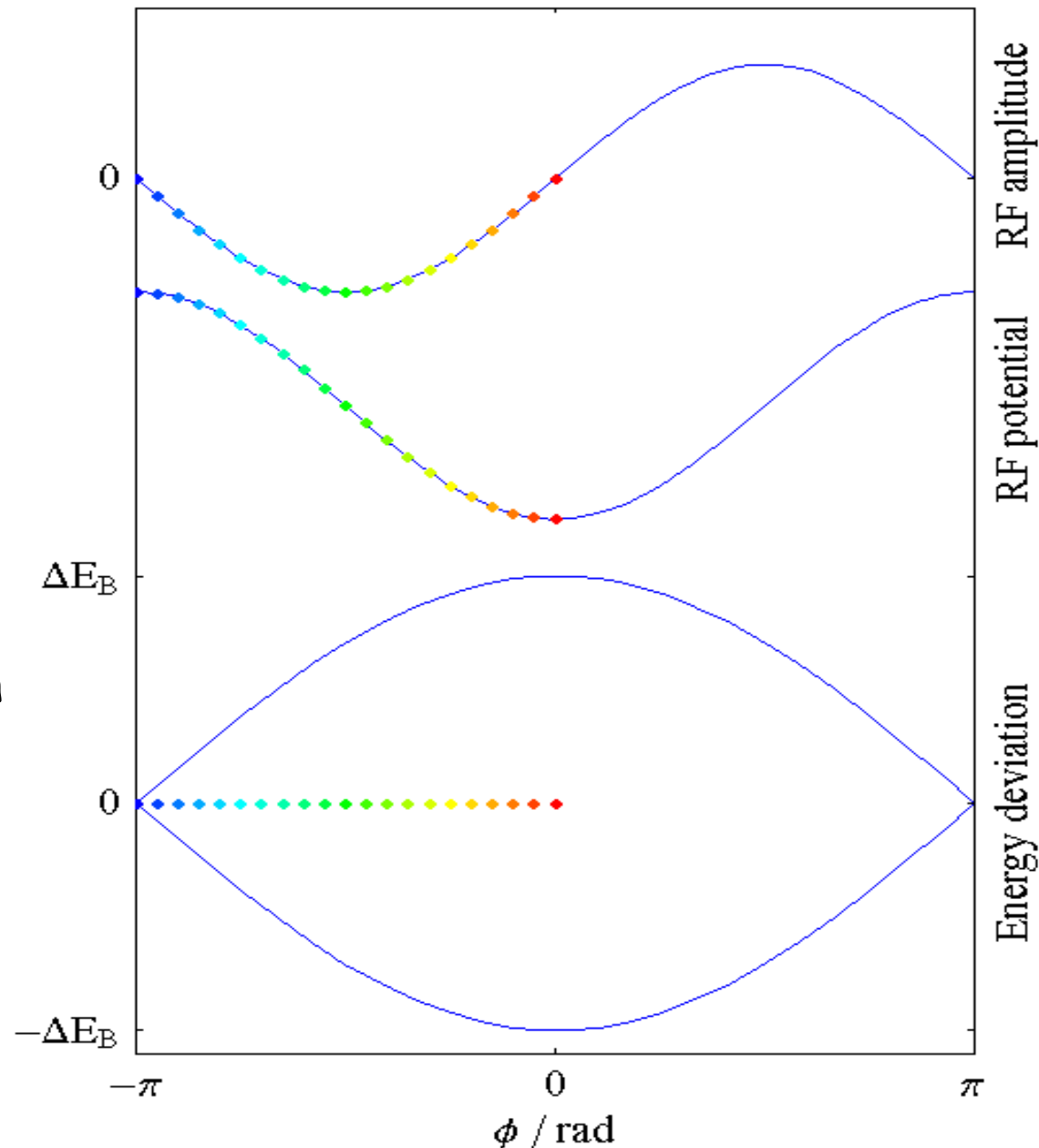
Attention: Different definitions are used!

Synchrotron motion in phase space

Remark:
Synchrotron frequency
much smaller than
betatron frequency.

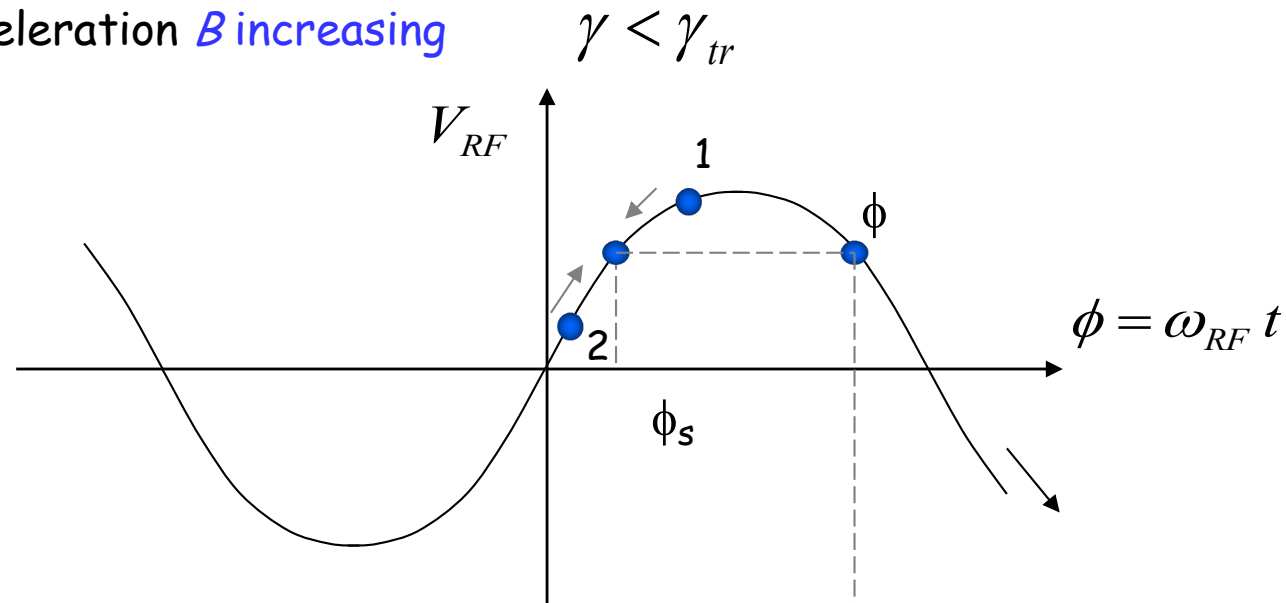
The restoring **force** is
non-linear.
 \Rightarrow speed of motion
depends on position in
phase-space

(here shown for a
stationary bucket)

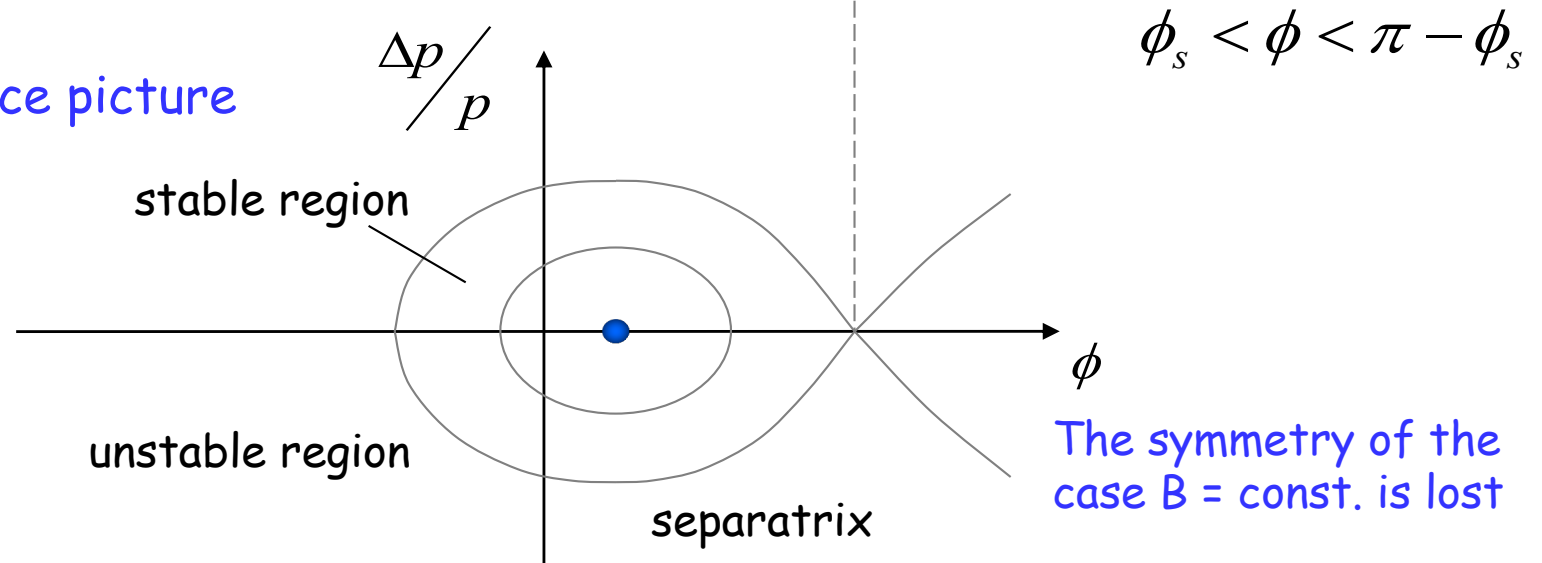


Synchrotron oscillations (with acceleration)

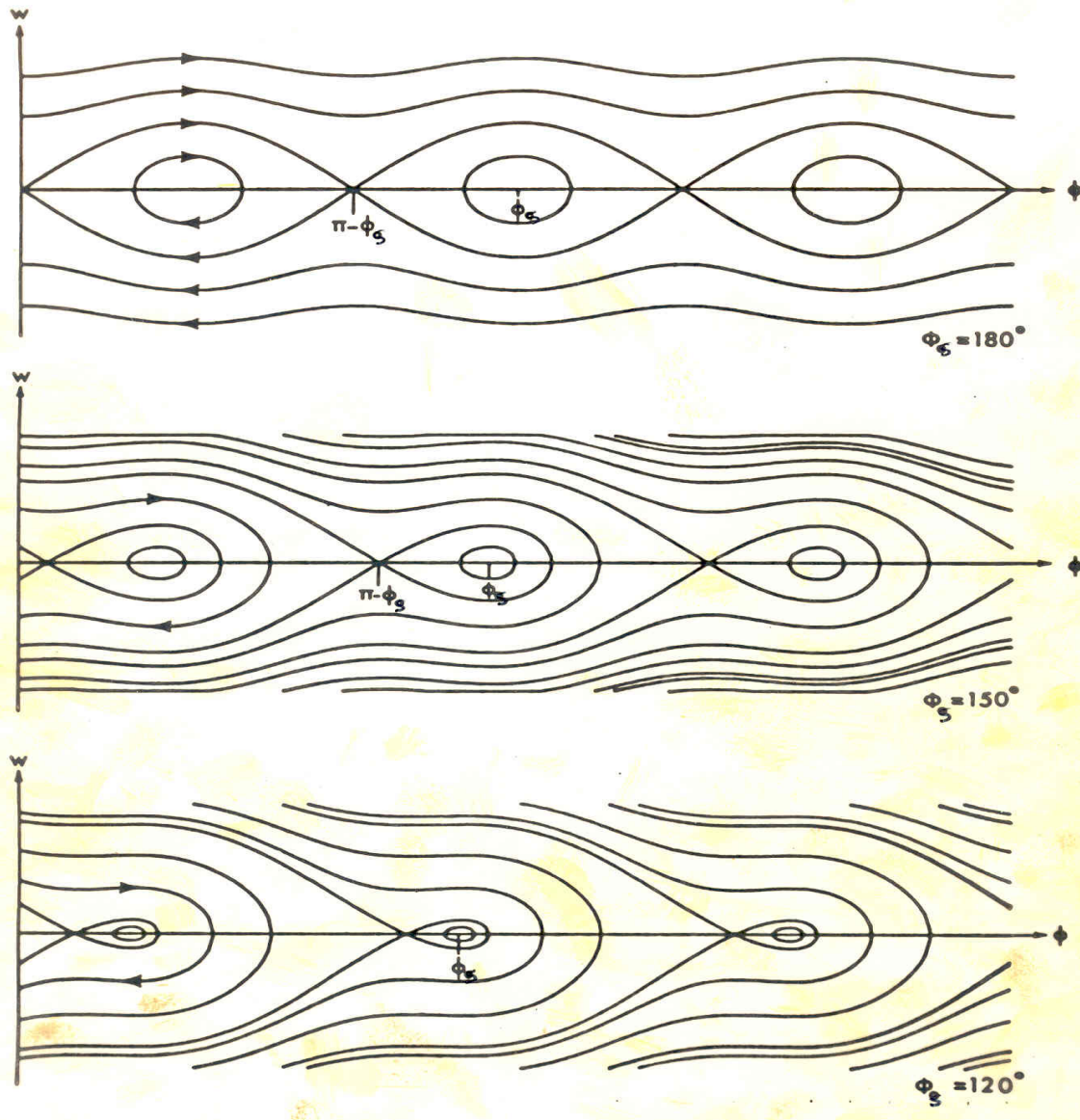
Case with acceleration B increasing



Phase space picture



RF Acceptance versus Synchronous Phase



The **areas of stable motion** (closed trajectories) are called "**BUCKET**". The number of circulating buckets is equal to " h ".

The phase extension of the **bucket is maximum** for $\phi_s = 180^\circ$ (or 0°) which means **no acceleration**.

During **acceleration**, the buckets get **smaller**, both in length and **energy acceptance**.

=> **Injection** preferably **without acceleration**.

Longitudinal Dynamics in Synchrotrons

Now we will look more quantitatively at the “**synchrotron motion**”.

The RF acceleration process clearly emphasizes two coupled variables, the **energy** gained by the particle and the **RF phase** experienced by the same particle.

Since there is a **well defined synchronous particle** which has always the same **phase** ϕ_s , and the nominal **energy** E_0 , it is sufficient to follow other particles with respect to that particle.

So let's introduce the following **reduced variables**:

particle RF phase	:	$\Delta\phi = \phi - \phi_s$
particle momentum	:	$\Delta p = p - p_0$
particle energy	:	$\Delta E = E - E_0$
angular frequency	:	$\Delta\omega = \omega - \omega_0$
azimuth orbital angle:		$\Delta\theta = \theta - \theta_s$

Look at **difference**
from synchronous
particle

Equations of Longitudinal Motion

In these reduced variables, the **equations of motion** are (see Appendix):

$$\frac{\Delta E}{\omega_0} = \frac{p_0 R}{h\eta\omega_0} \dot{\phi}$$

$$2\pi \frac{d}{dt} \left(\frac{\Delta E}{\omega_0} \right) = e \hat{V} (\sin \phi - \sin \phi_s)$$

deriving and combining

$$\frac{d}{dt} \left[\frac{-p_0 R}{h\eta\omega_0} \frac{d\phi}{dt} \right] + \frac{e \hat{V}}{2\pi} (\sin \phi - \sin \phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time.

We will study some cases in the following...

Hamiltonian of Longitudinal Motion

Introducing a new convenient variable, W , leads to the 1st order equations:

$$W = \frac{\Delta E}{\omega_0} \longrightarrow \begin{aligned} \frac{d\phi}{dt} &= \frac{h\eta\omega_0}{p_0 R} W \\ \frac{dW}{dt} &= \frac{e\hat{V}}{2\pi} (\sin \phi - \sin \phi_s) \end{aligned}$$

The two variables ϕ, W are canonical since these equations of motion can be derived from a Hamiltonian $H(\phi, W, t)$:

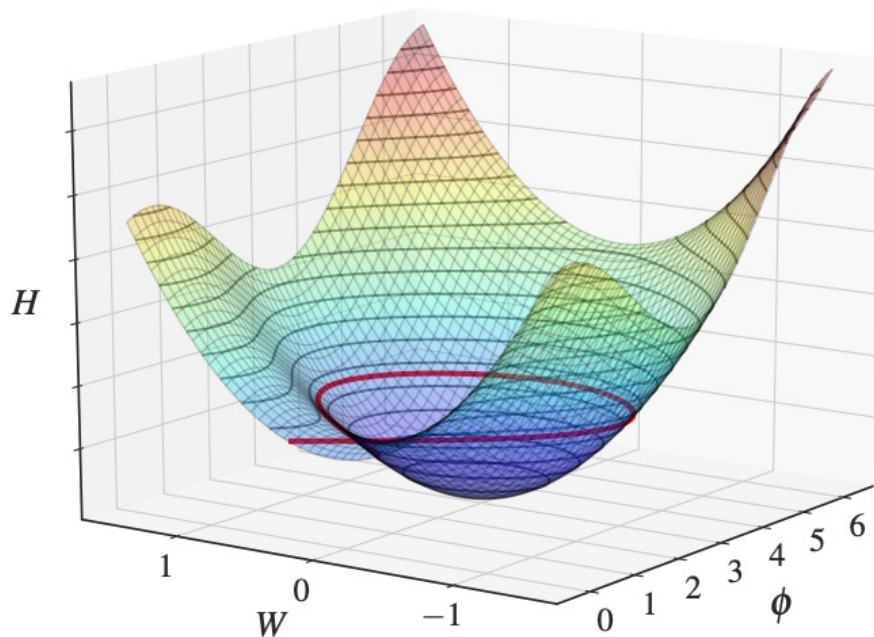
$$\frac{d\phi}{dt} = \frac{\partial H}{\partial W} \qquad \frac{dW}{dt} = -\frac{\partial H}{\partial \phi}$$

$$H(\phi, W) = \frac{1}{2} \frac{h\eta\omega_0}{p_0 R} W^2 + \frac{e\hat{V}}{2\pi} [\cos \phi - \cos \phi_s + (\phi - \phi_s) \sin \phi_s]$$

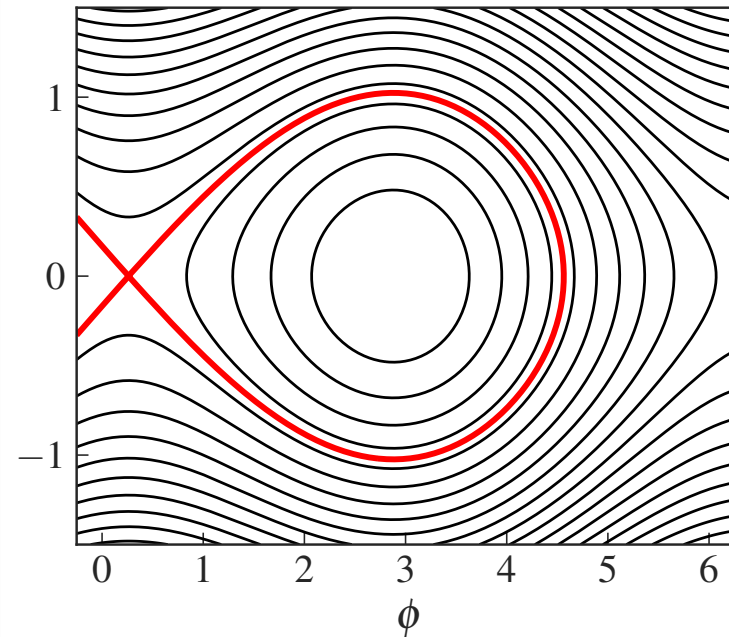
Hamiltonian of Longitudinal Motion

What does it represent? The total energy of the system!

Surface of $H(\phi, W)$



Contours of $H(\phi, W)$



Contours of constant H are particle trajectories in phase space!
(H is conserved)

Hamiltonian Mechanics can help us understand some fairly complicated dynamics (multiple harmonics, bunch splitting, ...)

Small Amplitude Oscillations

Let's assume constant parameters R , p_0 , ω_0 and η :

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos \phi_s} (\sin \phi - \sin \phi_s) = 0$$

with

$$\Omega_s^2 = \frac{-q \hat{V}_{RF} \eta h \omega_0}{2\pi R p_0} \cos \phi_s$$

Consider now **small** phase **deviations** from the reference particle:

$$\sin \phi - \sin \phi_s = \sin(\phi_s + \Delta\phi) - \sin \phi_s \cong \cos \phi_s \Delta\phi \quad (\text{for small } \Delta\phi)$$

and the corresponding linearized motion reduces to a **harmonic oscillation**:

$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0 \quad \text{where } \Omega_s \text{ is the } \textbf{synchrotron angular frequency}.$$

The **synchrotron tune** ν_s is the number of synchrotron oscillations per revolution:

$$\nu_s = \Omega_s / \omega_0$$

Typical values are $\ll 1$, as it takes several 10 - 1000 turns per oscillation.

- proton synchrotrons of the order 10^{-3}
- electron storage rings of the order 10^{-1}

Stability condition for ϕ_s

$$\Omega_s^2 = \frac{-q\hat{V}_{RF}\eta h\omega_0}{2\pi R p_0} \cos \phi_s \Leftrightarrow \Omega_s^2 = \omega_0^2 \frac{-q\hat{V}_{RF}\eta h}{2\pi\beta^2 E} \cos \phi_s \quad \text{with } Rp = \frac{\beta^2 E}{\omega}$$

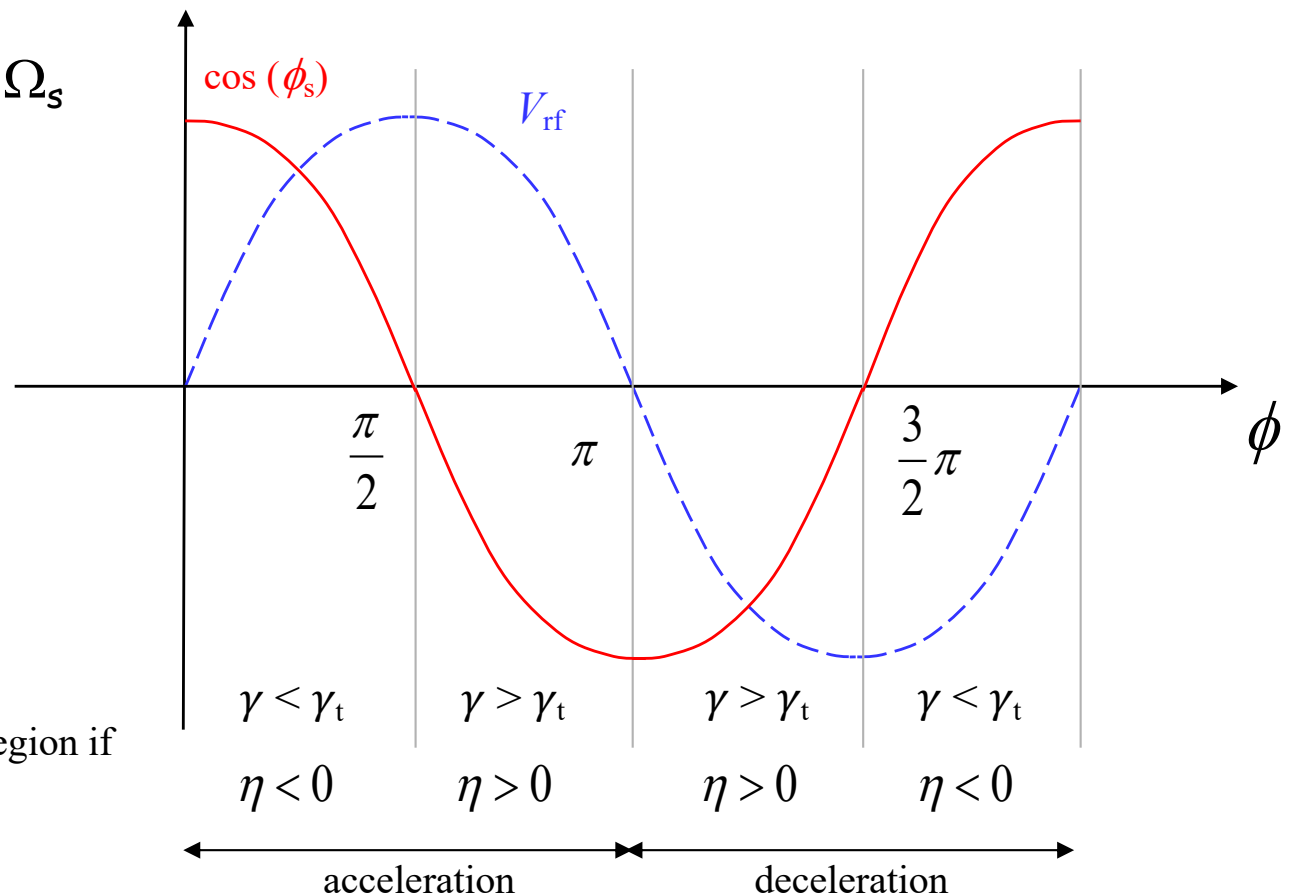
Stability is obtained when Ω_s is real and so Ω_s^2 positive:

$$\Omega_s^2 > 0$$

\Leftrightarrow

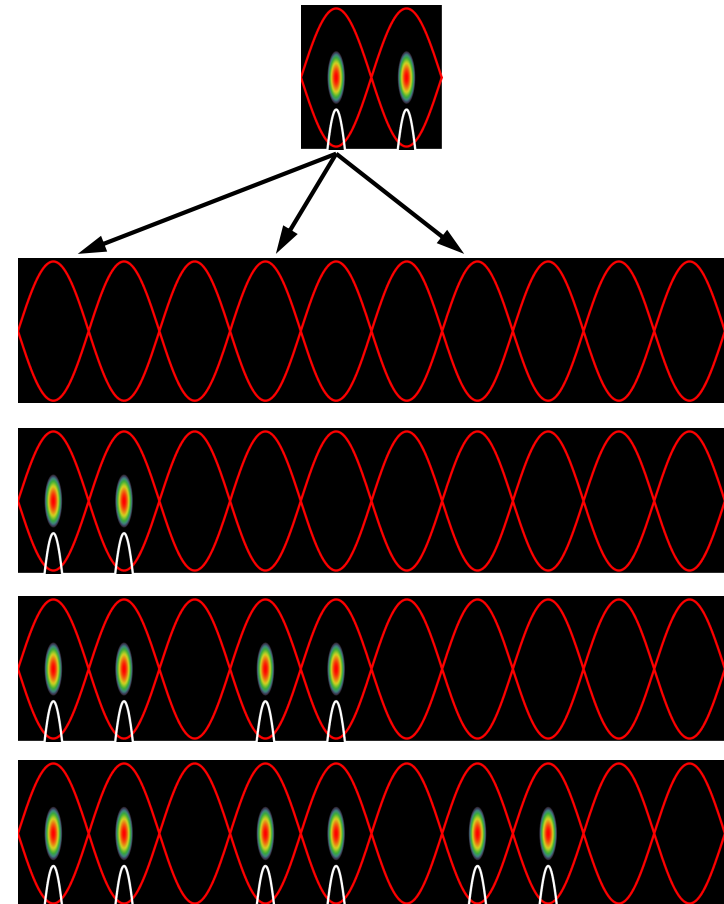
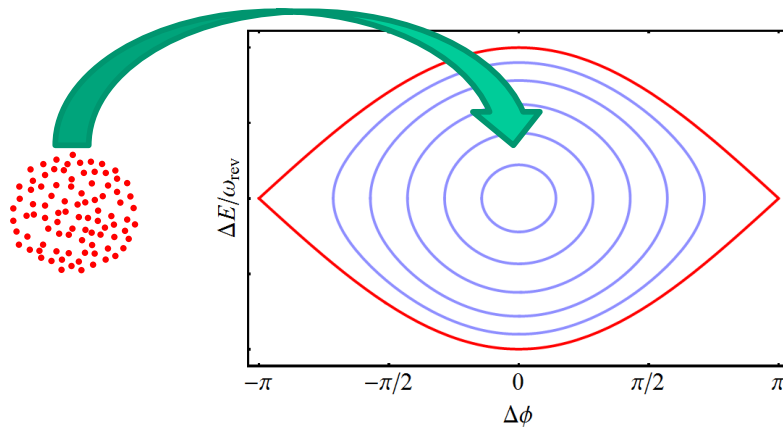
$$\eta \cos \phi_s < 0$$

Stable in the region if



Bunch-to-bucket transfer

- Bunch from sending accelerator into the bucket of receiving



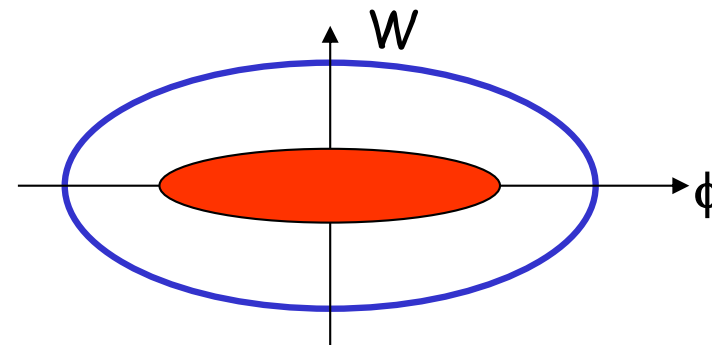
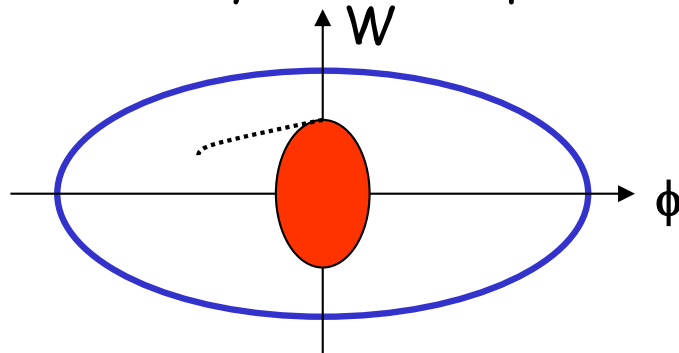
Advantages:

- Particles always subject to longitudinal focusing
- No need for RF capture of de-bunched beam in receiving accelerator
- No particles at unstable fixed point
- Time structure of beam preserved during transfer

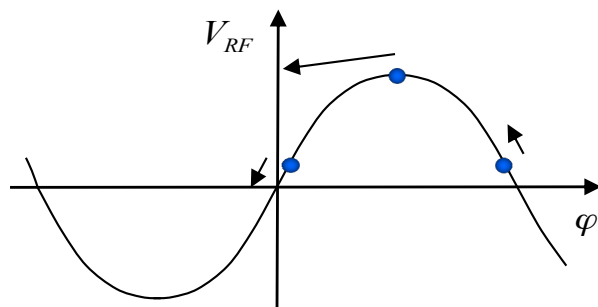
Bunch Transfer - Effect of a Mismatch

When you **transfer** the bunch from one RF system to another, the shape of the **phase space and the bunch need to match**.

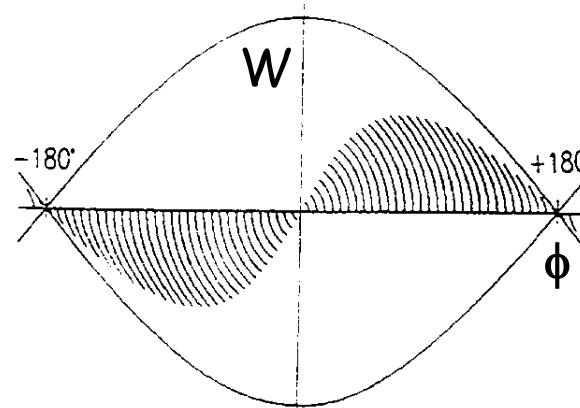
Mismatch example: Injected bunch: short length and large energy spread
after 1/4 synchrotron period: longer bunch with a smaller energy spread.



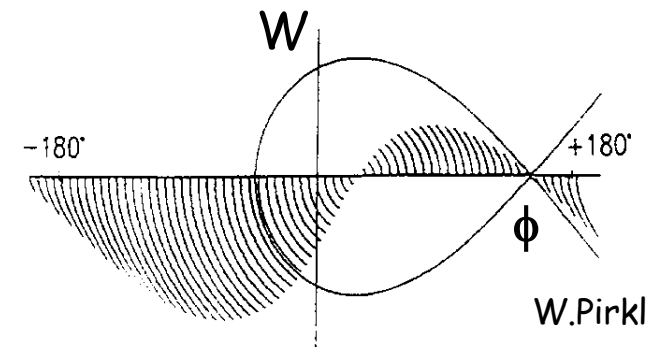
For **larger amplitudes**, the angular phase space motion is slower (1/8 period shown below) => can lead to **filamentation** and **emittance growth**



restoring force is
non-linear



stationary bucket



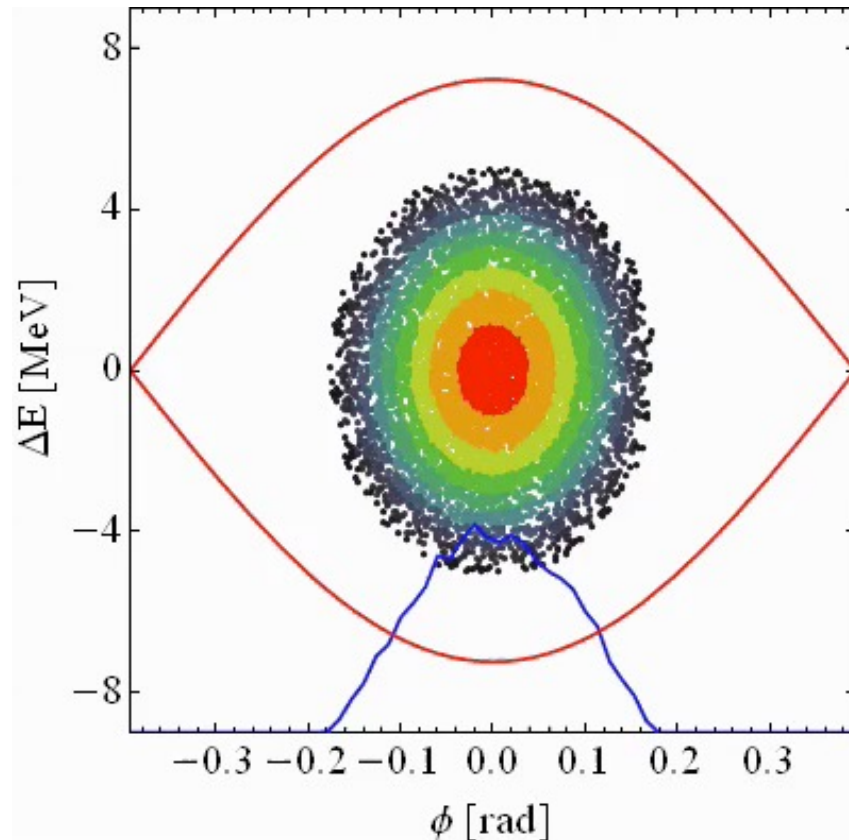
accelerating bucket

W.Pirkl

Effect of a Mismatch (2)

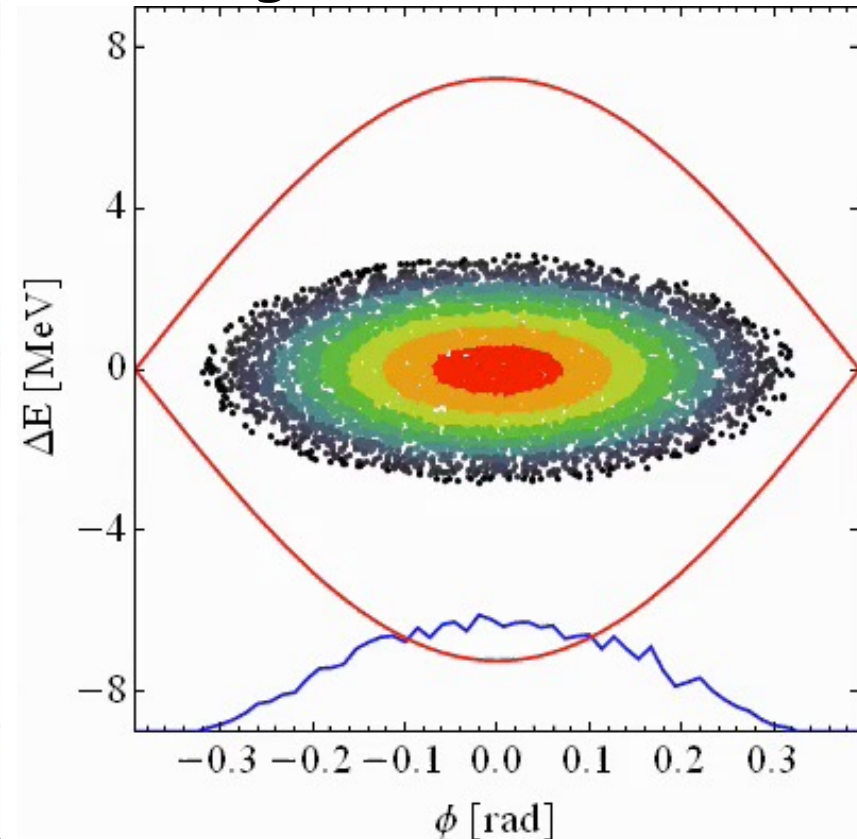
- Long. emittance is only preserved for **correct RF voltage**

Matched case



→ Bunch is fine, longitudinal emittance remains constant

Longitudinal mismatch

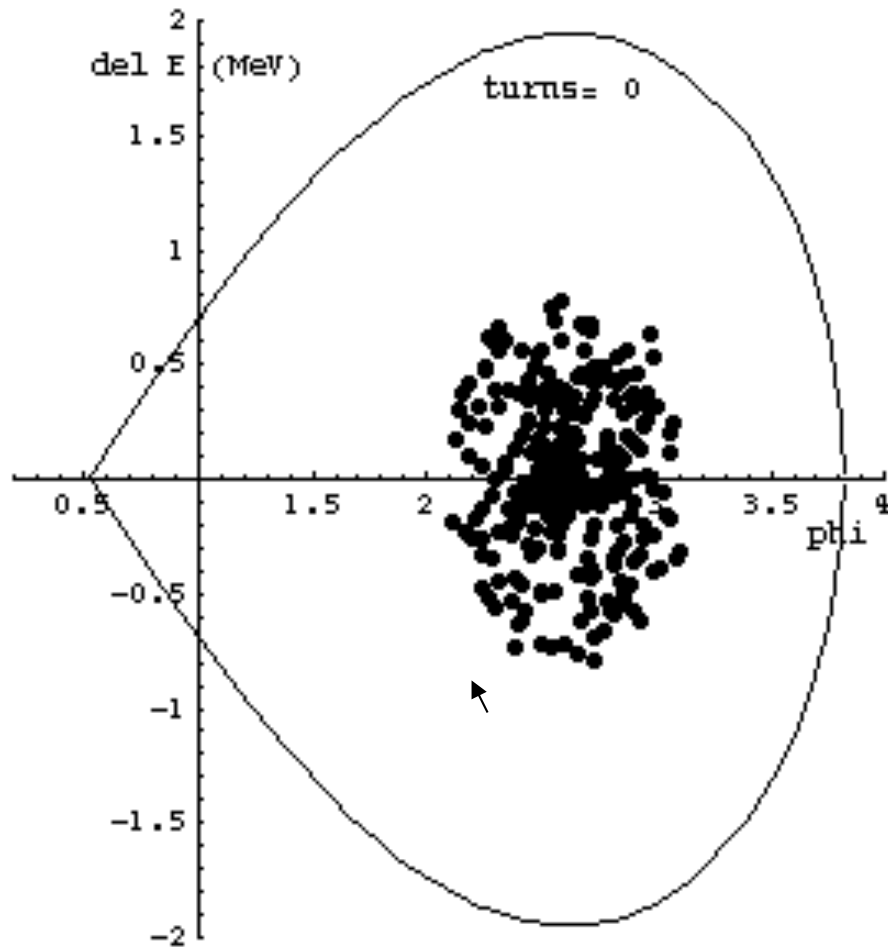


→ Dilution of bunch results in increase of long. emittance

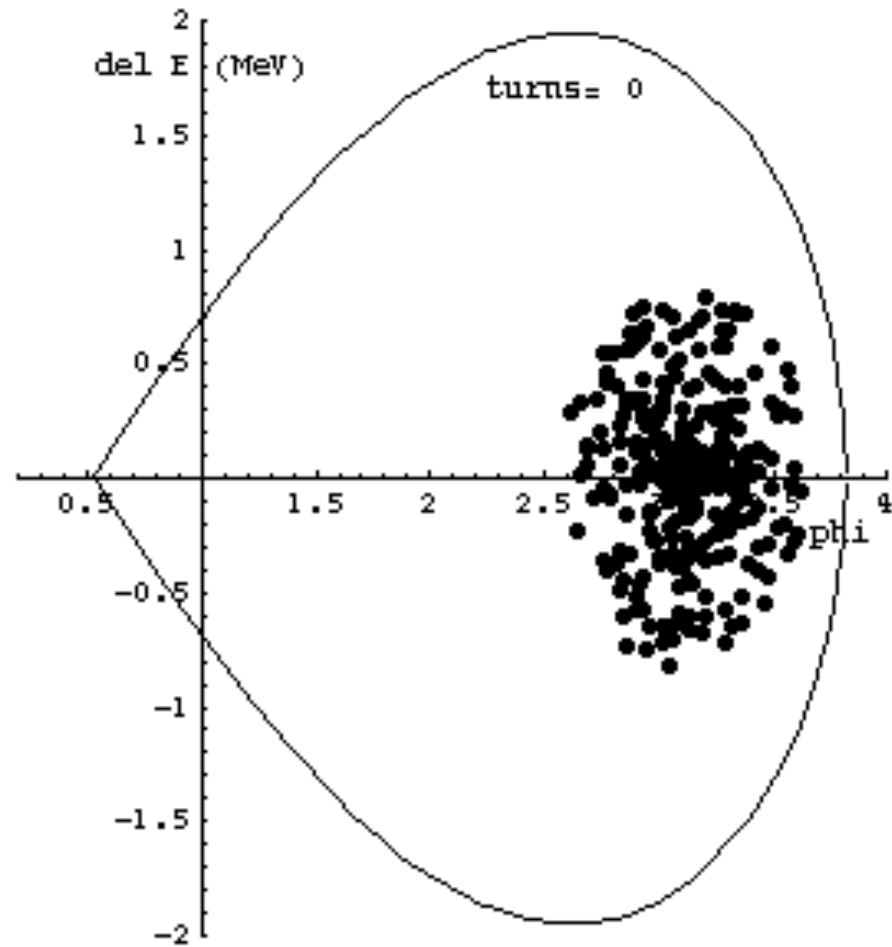
Effect of a Mismatch (3)

Evolution of an injected beam for the first 100 turns.

For a mismatched transfer, the emittance increases (right).



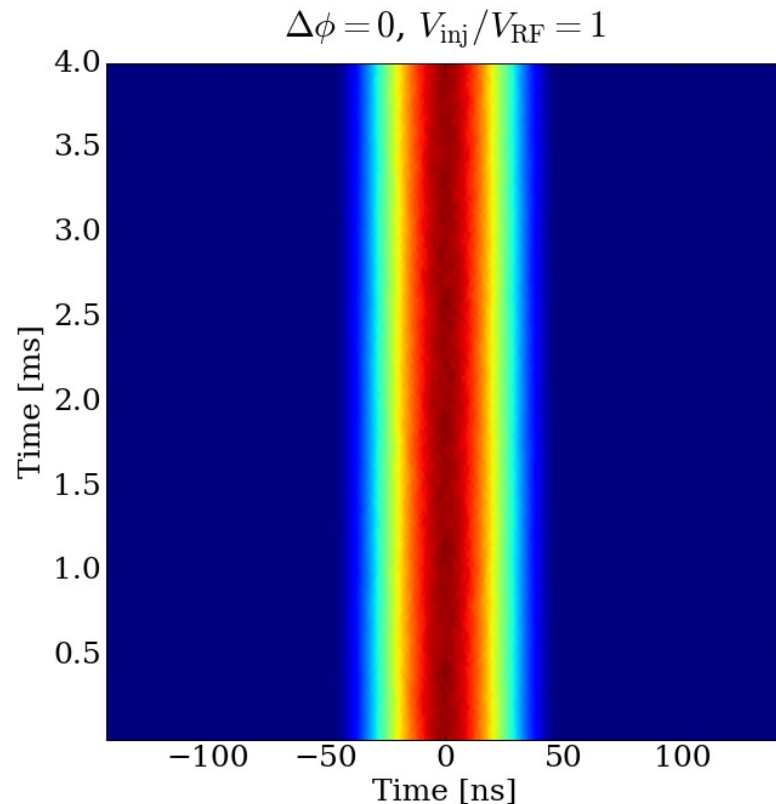
matched beam



mismatched beam - **phase error**

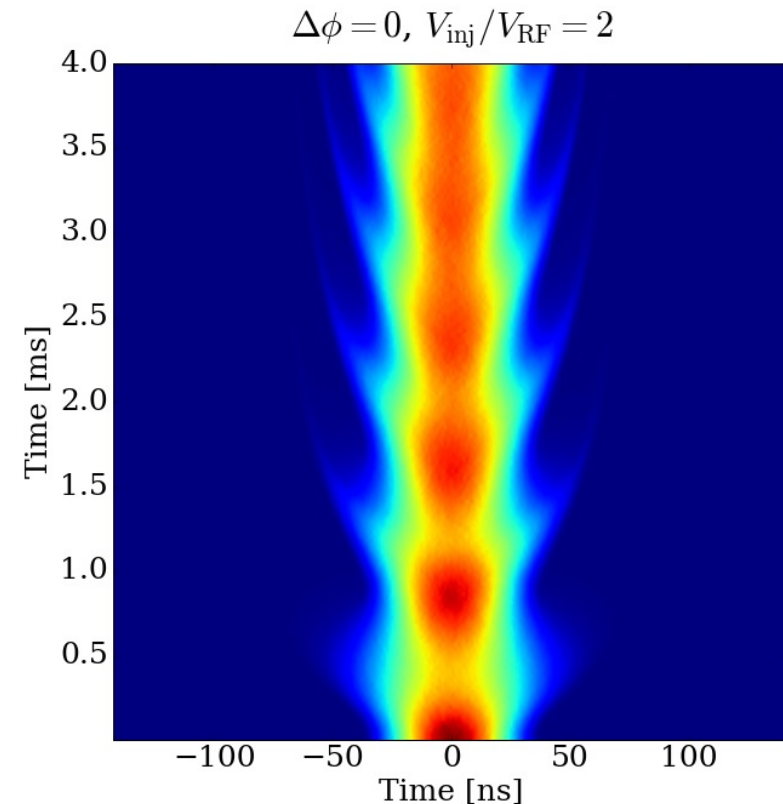
Longitudinal matching - Beam profile

Matched case



→ Bunch is fine, longitudinal emittance remains constant

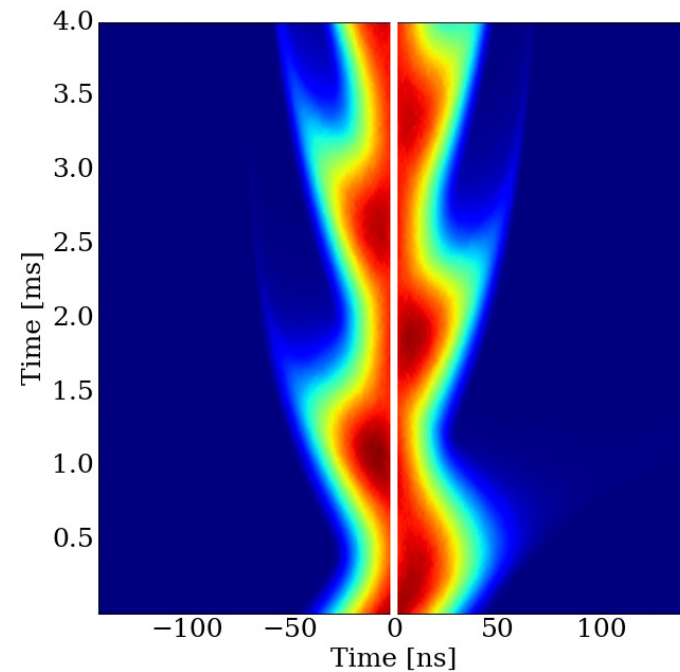
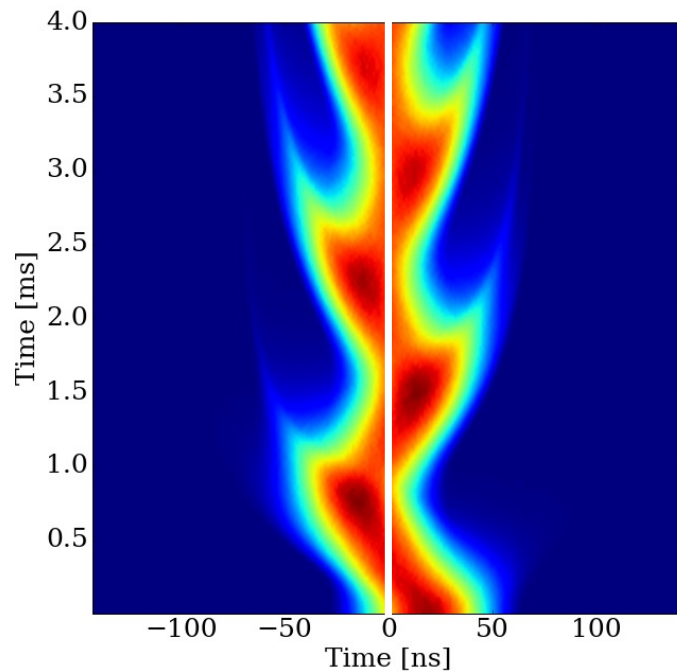
Longitudinal mismatch



→ Dilution of bunch results in increase of long. emittance

Matching quiz!

- Find the difference!



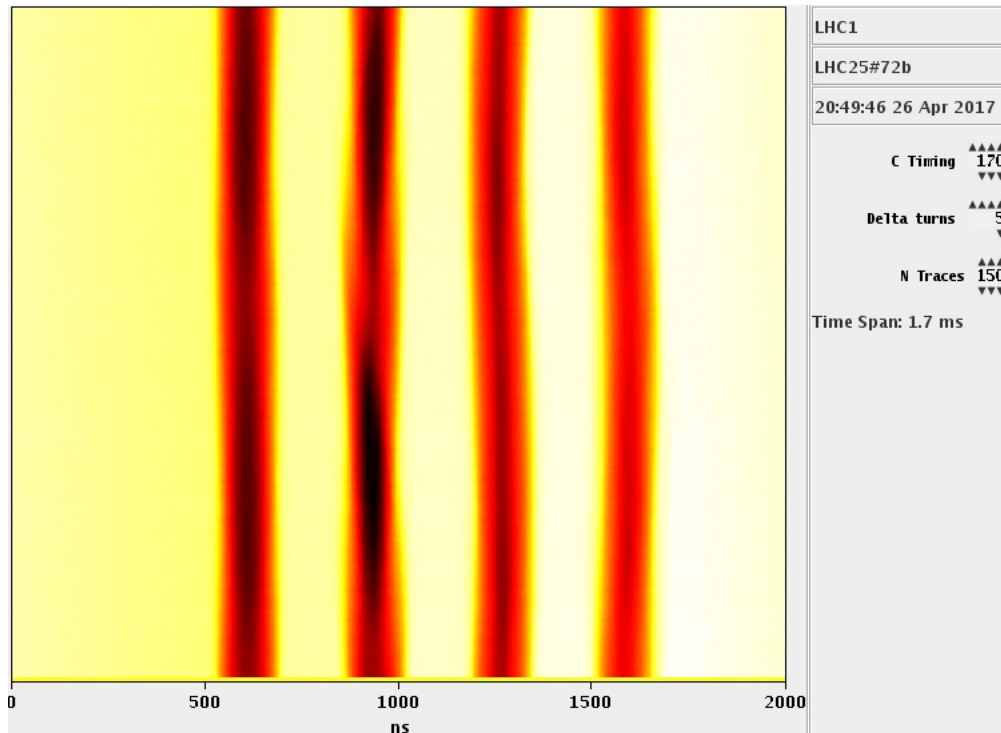
- -45° phase error at injection
- Can be easily corrected by bucket phase

- Equivalent energy error
- Phase does not help: requires beam energy change

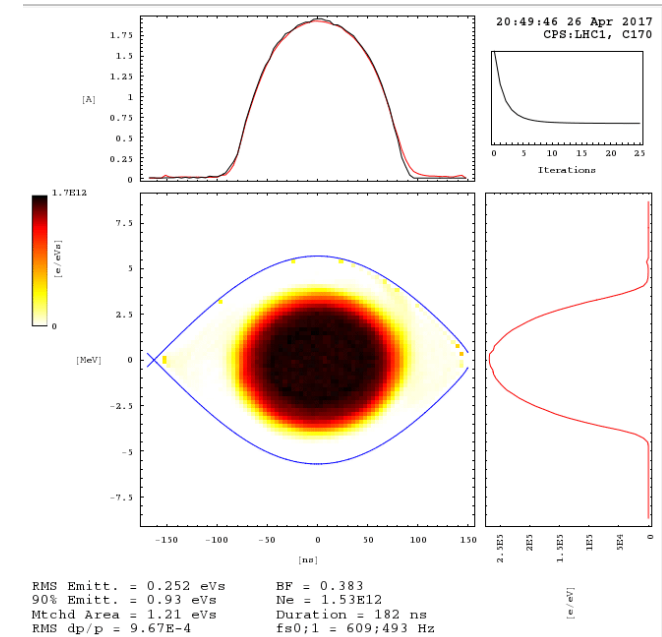
Phase Space Tomography

We can reconstruct the phase space distribution of the beam.

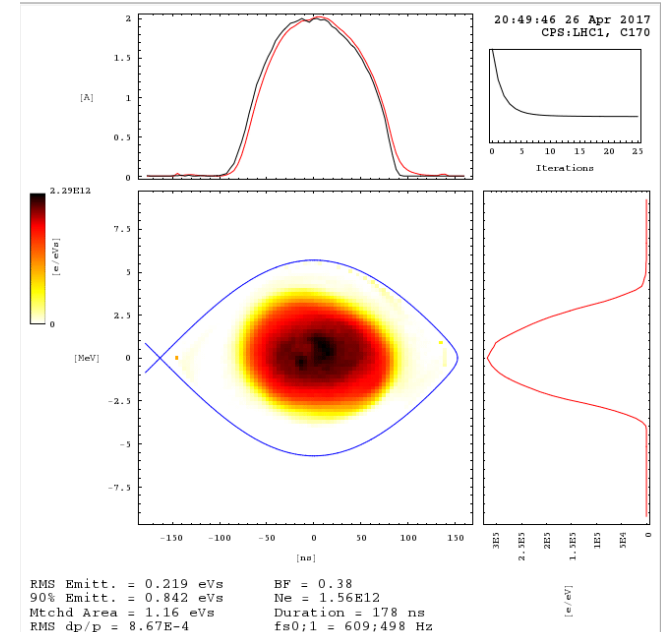
- Longitudinal bunch profiles over a number of turns
- Parameters determining Ω_s



1st
bunch



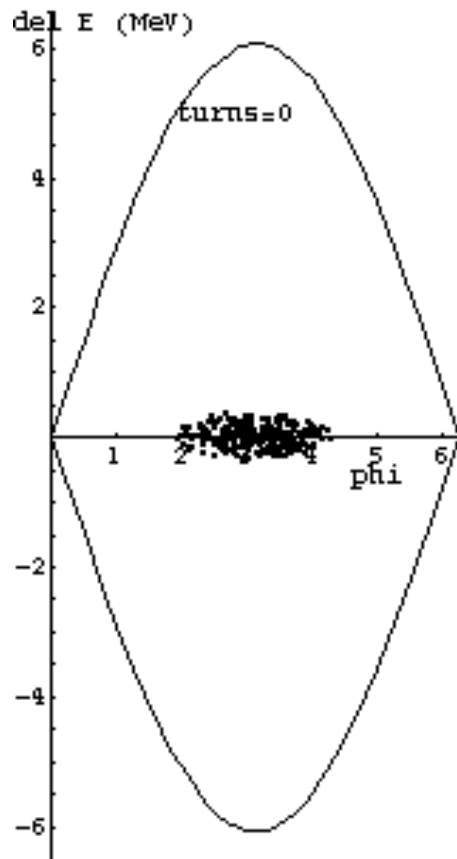
2nd
bunch



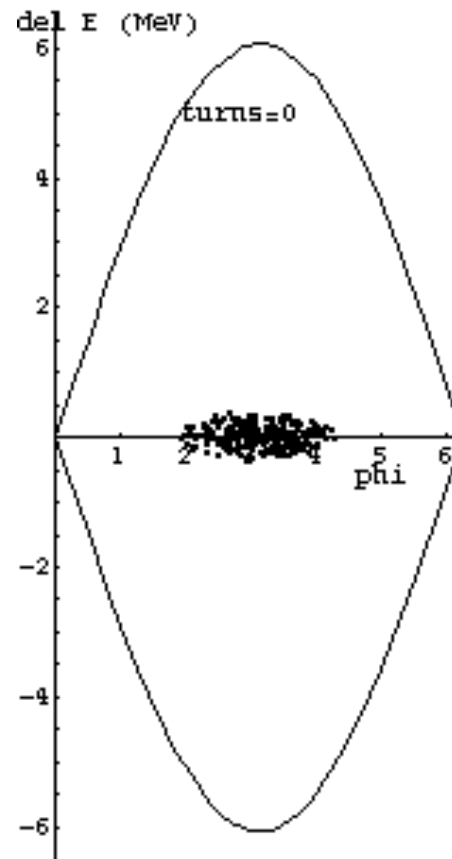
Bunch Rotation

Phase space motion can be used to make short bunches.

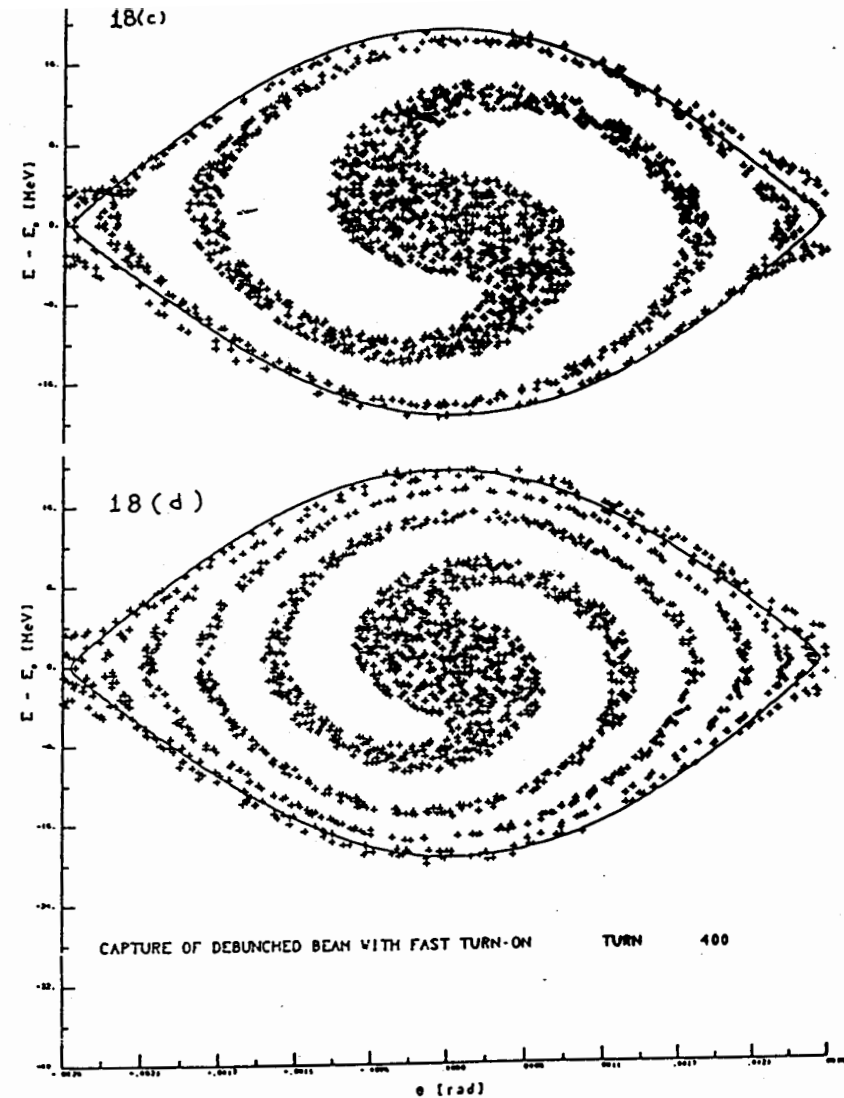
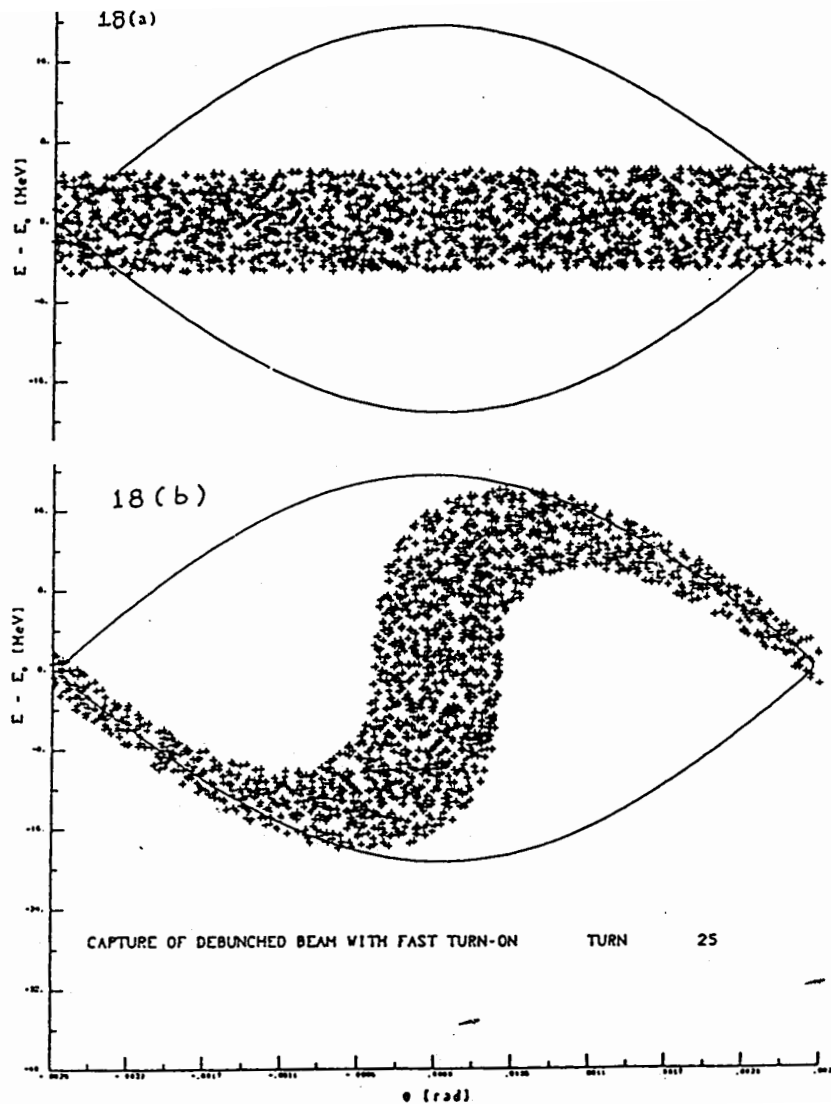
Start with a long bunch and extract or recapture when it's short.



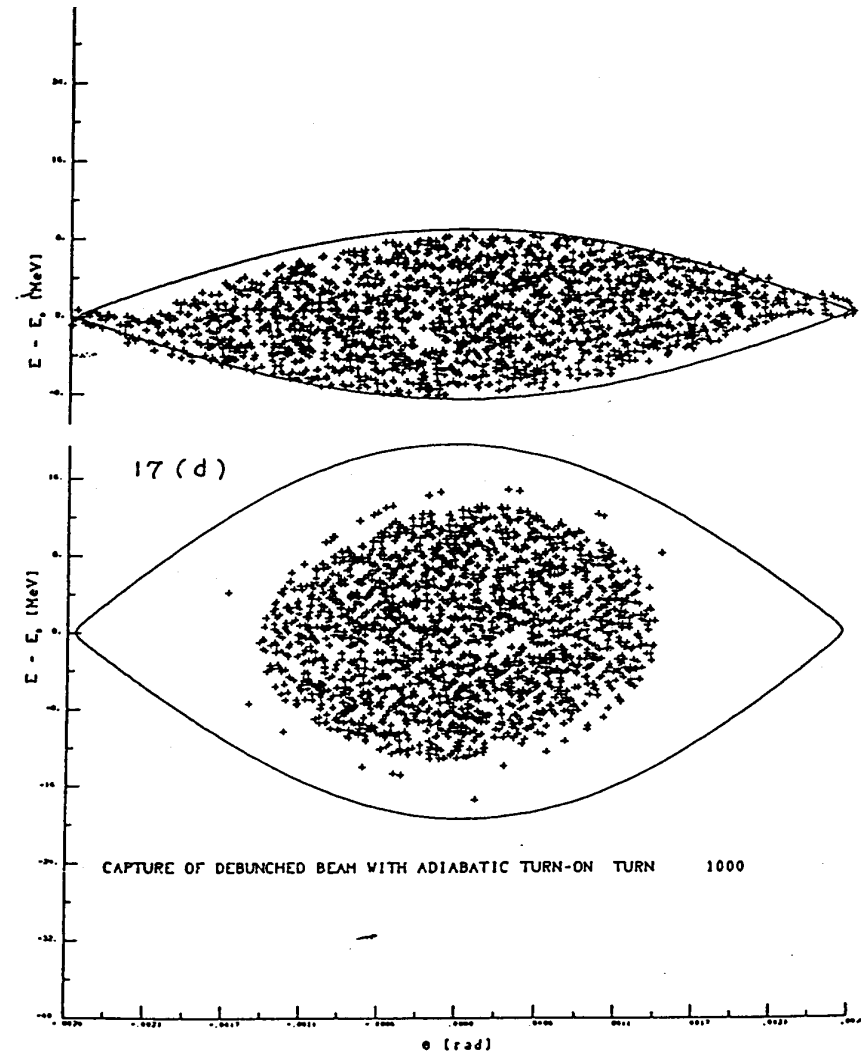
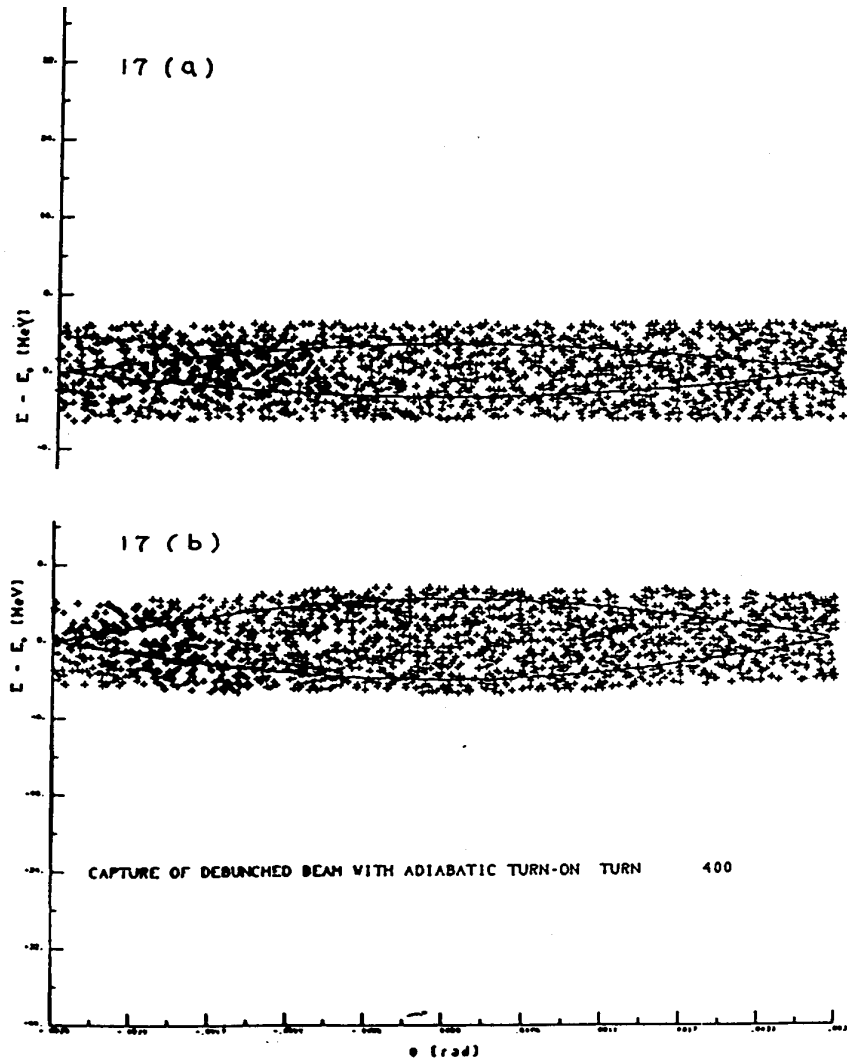
initial beam



Capture of a Debunched Beam with Fast Turn-On

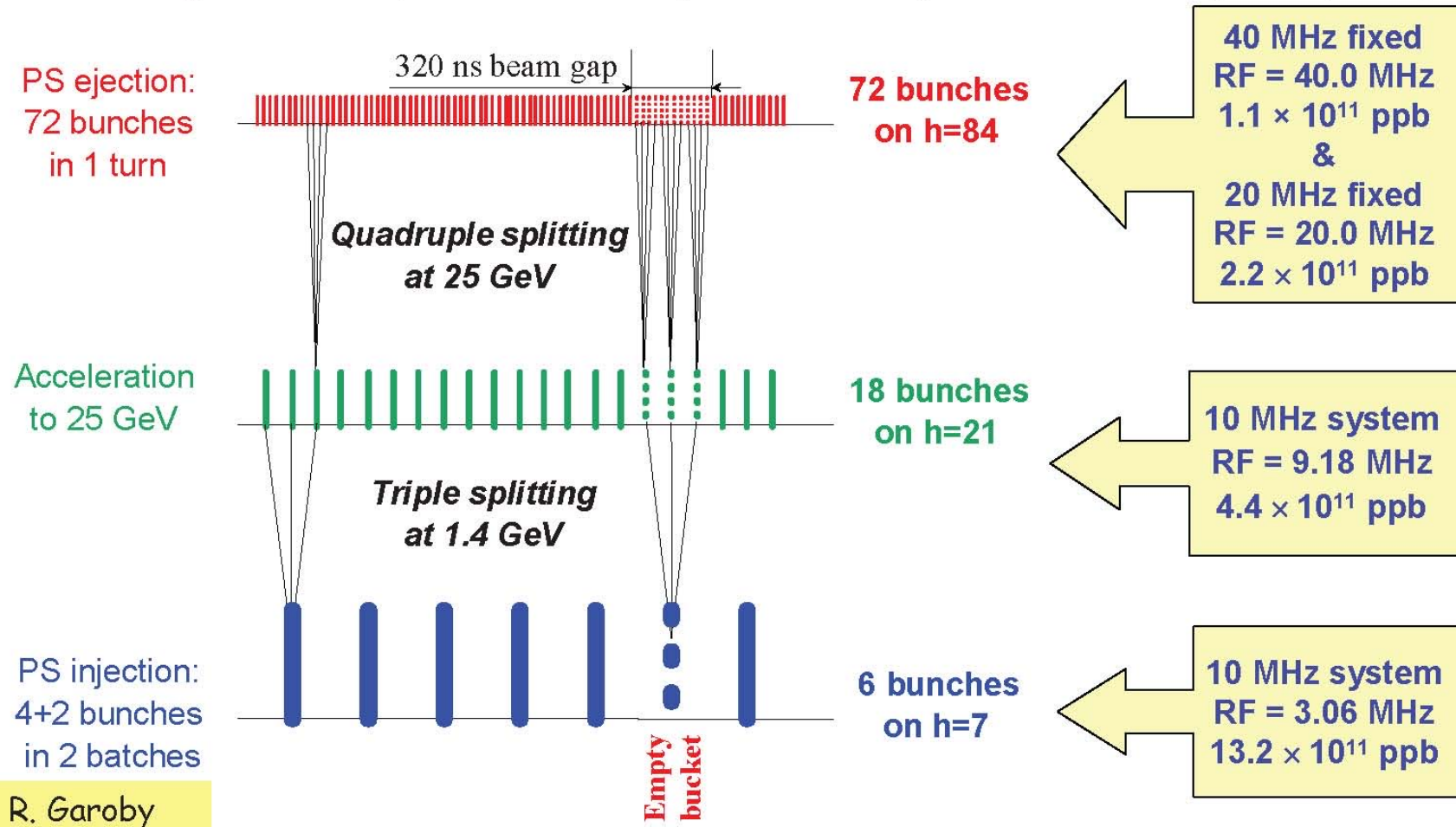


Capture of a Debunched Beam with Adiabatic Turn-On



Generating a 25ns LHC Bunch Train in the PS

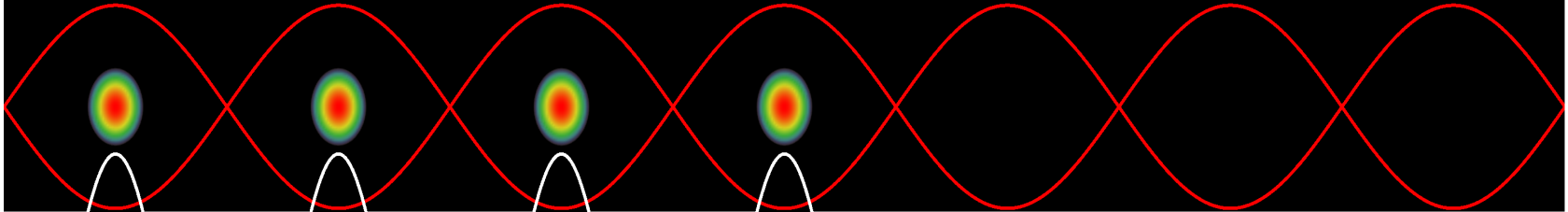
- **Longitudinal bunch splitting (basic principle)**
 - Reduce voltage on principal RF harmonic and simultaneously rise voltage on multiple harmonics (adiabatically with correct phase, etc.)



Use double splitting at 25 GeV to generate 50ns bunch trains instead

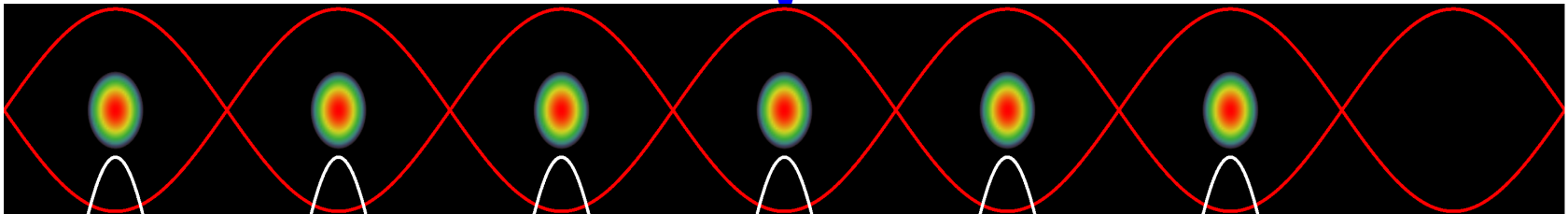
Production of the LHC 25 ns beam

1. Inject four bunches ~ 180 ns, 1.3 eVs

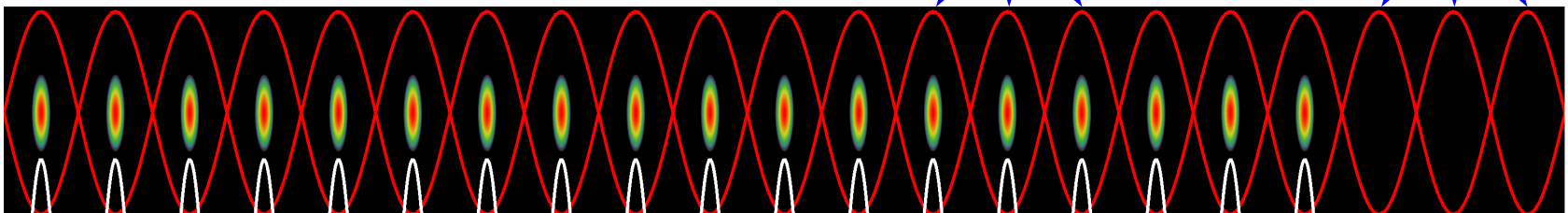


Wait 1.2 s for second injection

2. Inject two bunches



3. Triple split after second injection

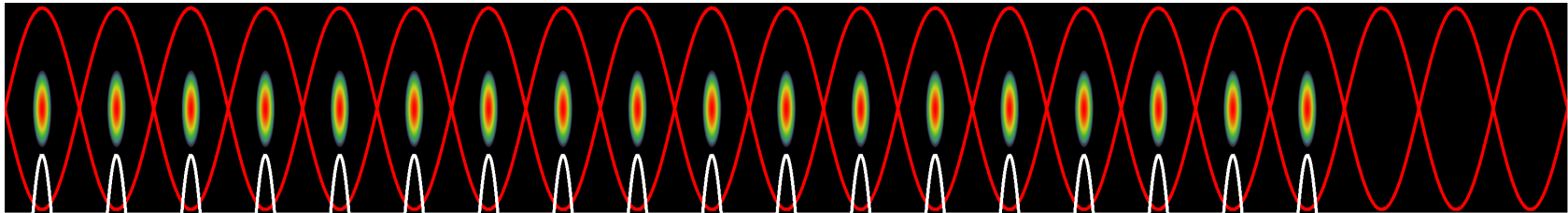


~ 0.7 eVs

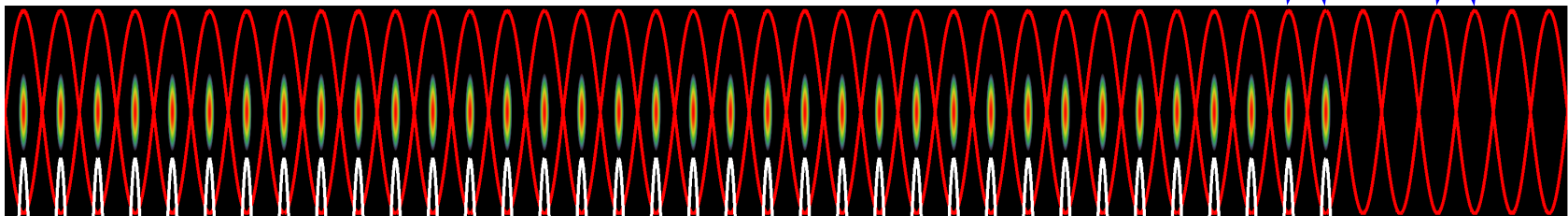
4. Accelerate from 1.4 GeV (E_{kin}) to 26 GeV

Production of the LHC 25 ns beam

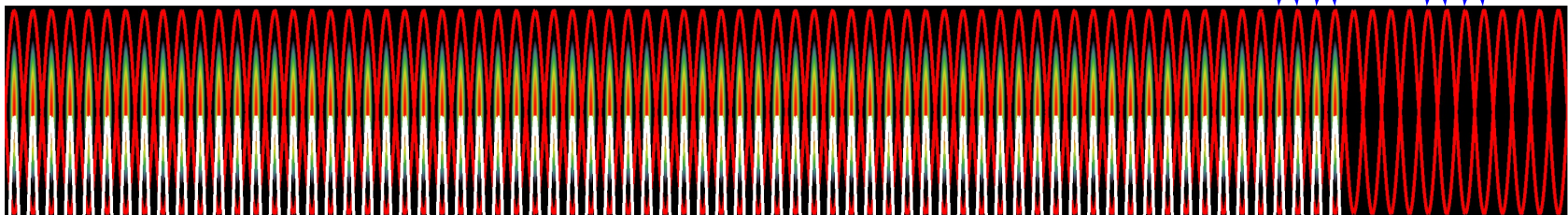
5. During acceleration: longitudinal emittance blow-up: **0.7 – 1.3 eVs**



6. Double split ($h_{21} \rightarrow h_{42}$)

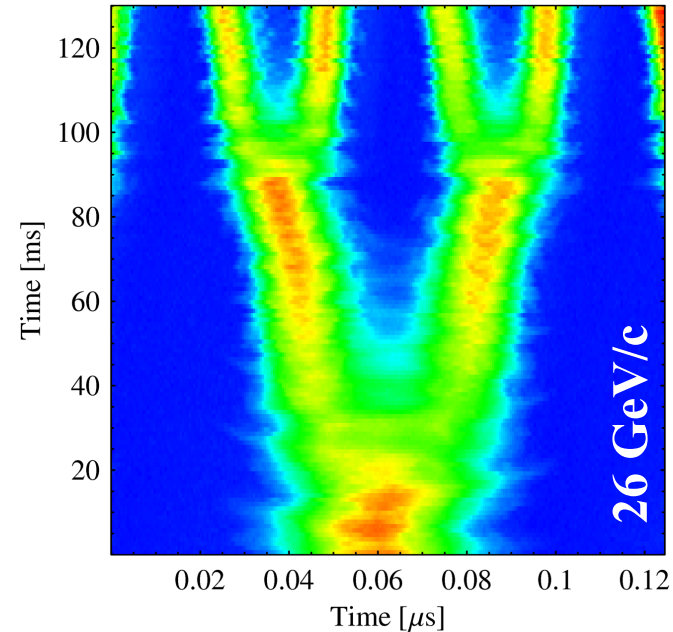
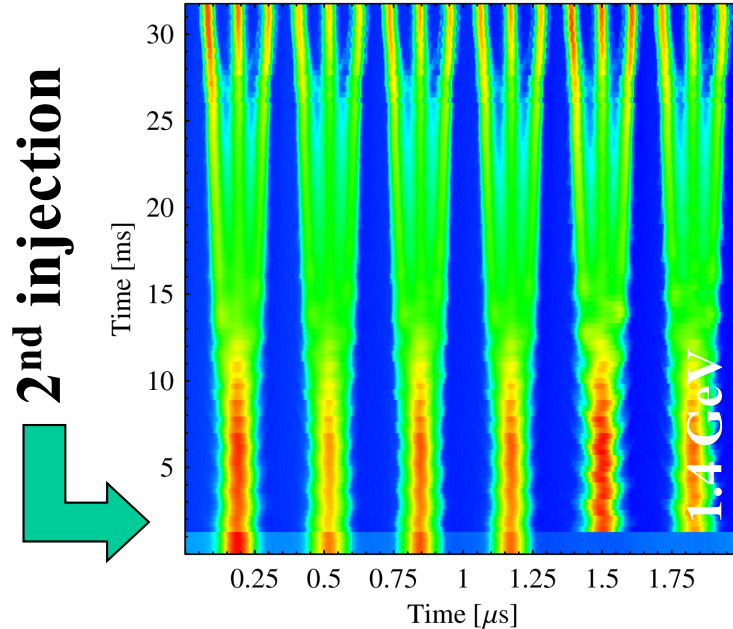
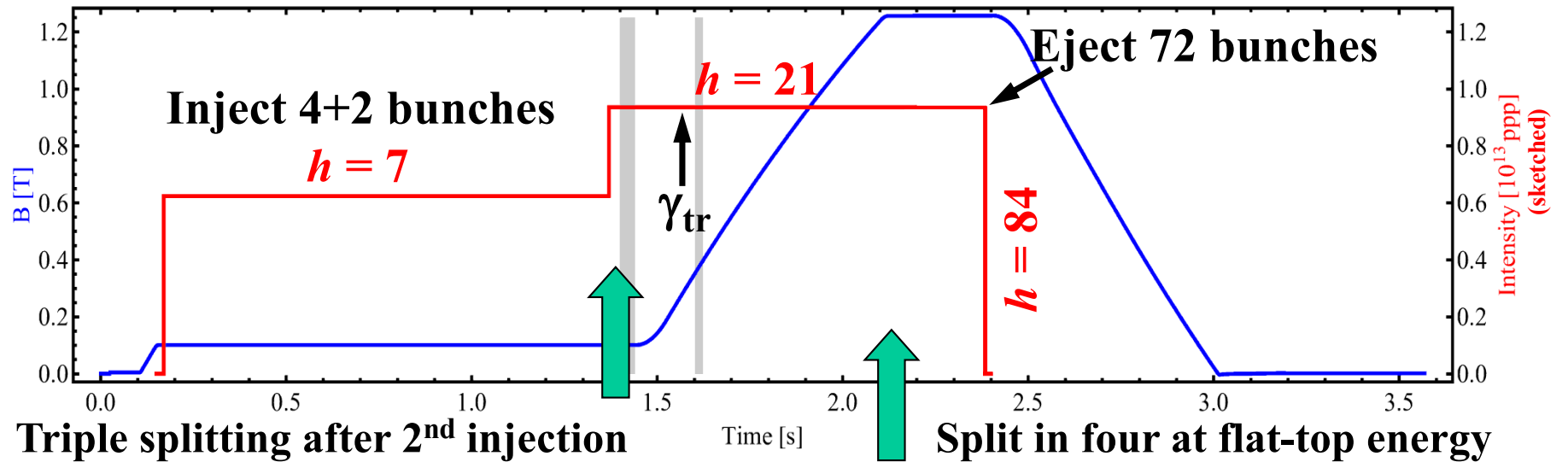


7. Double split ($h_{42} \rightarrow h_{84}$) ~ 0.35 eVs, 4 ns



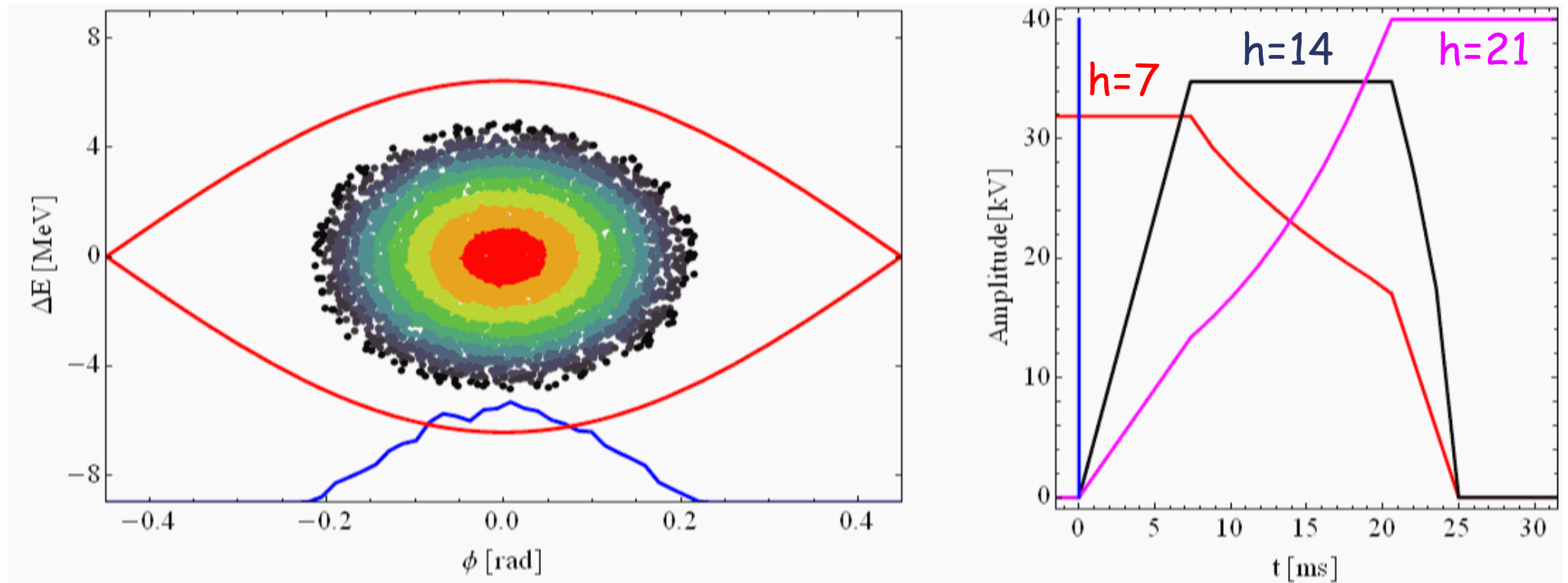
10. Fine synchronization, bunch rotation \rightarrow Extraction!

The LHC25 (ns) cycle in the PS



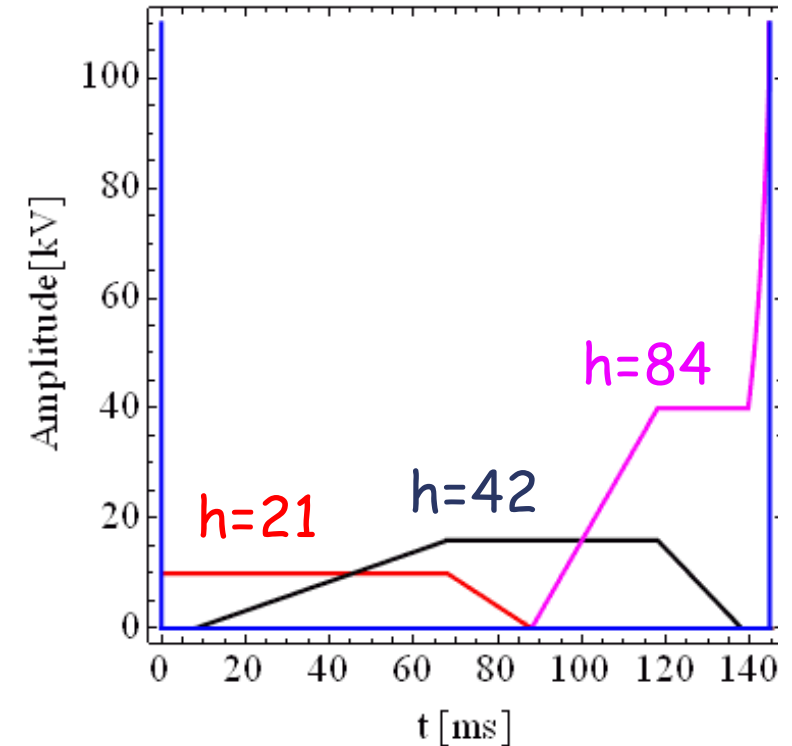
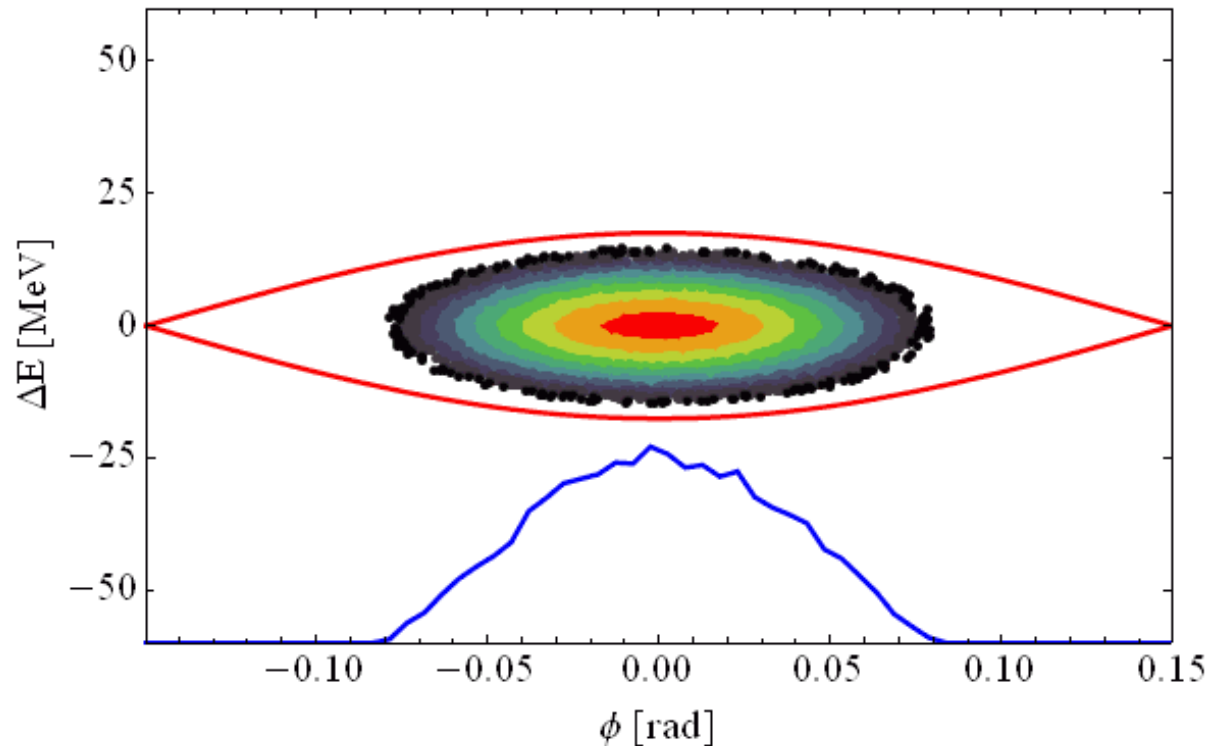
→ Each bunch from the Booster divided by 12 → $6 \times 3 \times 2 \times 2 = 72$

Triple splitting in the PS



Two times double splitting in the PS

Two times double splitting and bunch rotation:



- Bunch is divided twice using RF systems at $h = 21/42$ (10/20 MHz) and $h = 42/84$ (20/40 MHz)
- Bunch rotation: first part $h84$ only + $h168$ (80 MHz) for final part

Summary



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- | | |
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And CERN Accelerator Schools (CAS) Proceedings
In particular: <https://arxiv.org/abs/2011.02932>
Longitudinal Beam Dynamics in Circular Accelerators

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I would like to thank everyone for the material that I have used.

In particular (hope I don't forget anyone):

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- Mike Syphers
- Daniel Schulte
- Roberto Corsini
- Roland Garoby
- Chris Warsop

Velocity, Energy and Momentum

normalized velocity $\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$

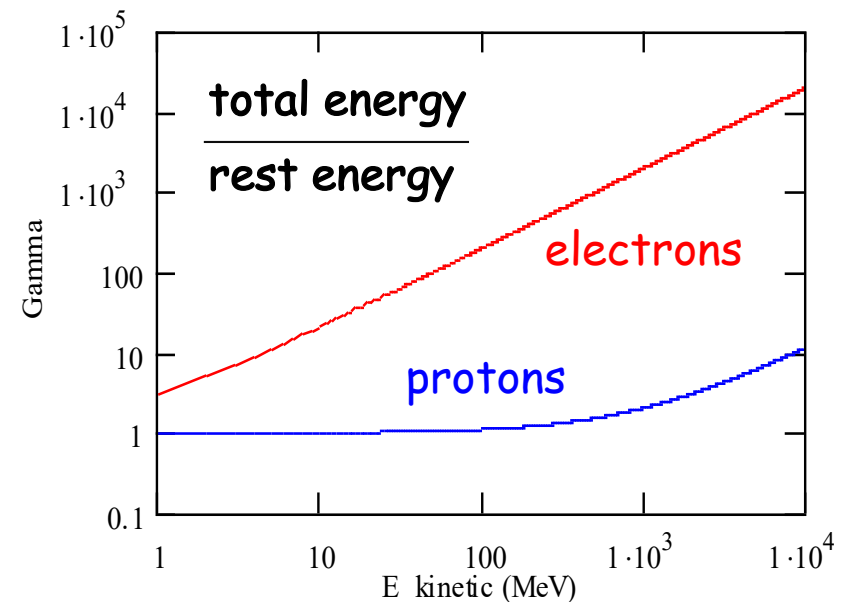
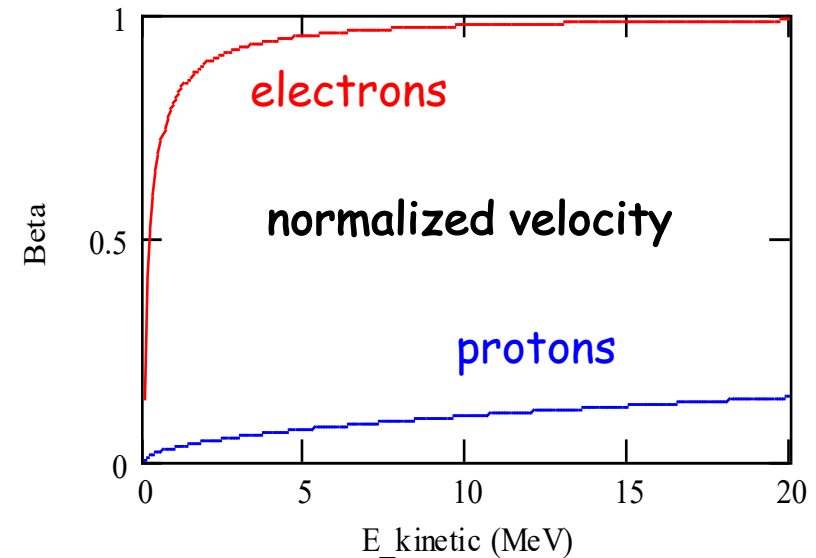
=> electrons almost reach the speed of light very quickly (few MeV range)

total energy
rest energy

$$E = \gamma m_0 c^2$$

$$\gamma = \frac{E}{E_0} = \frac{m}{m_0} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - \beta^2}}$$

Momentum $p = mv = \frac{E}{c^2} \beta c = \beta \frac{E}{c} = \beta \gamma m_0 c$



Derivation: Momentum Compaction Factor

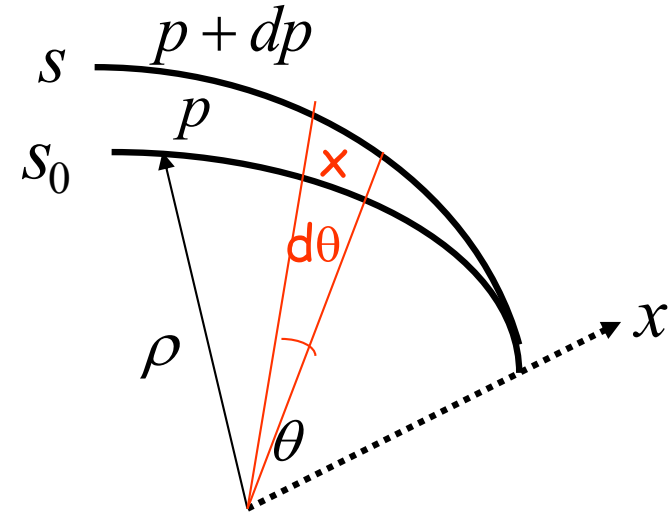
$$\alpha_c = \frac{p}{L} \frac{dL}{dp}$$

$$ds_0 = \rho d\theta$$

$$ds = (\rho + x) d\theta$$

The elementary path difference from the two orbits is:

$$\frac{dl}{ds_0} = \frac{ds - ds_0}{ds_0} = \frac{x}{\rho} \stackrel{\text{definition of dispersion } D_x}{=} \frac{D_x}{\rho} \frac{dp}{p}$$



leading to the total change in the circumference:

$$dL = \int_C dl = \int \frac{x}{\rho} ds_0 = \int \frac{D_x}{\rho} \frac{dp}{p} ds_0$$

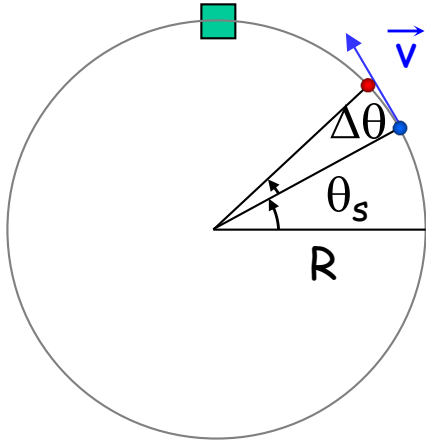
$$\alpha_c = \frac{1}{L} \int_C \frac{D_x(s)}{\rho(s)} ds_0$$

With $\rho = \infty$ in straight sections we get:

$$\alpha_c = \frac{\langle D_x \rangle_m}{R}$$

$\langle \rangle_m$ means that the average is considered over the bending magnet only

Appendix: First Energy-Phase Equation



$$f_{RF} = h f_r \Rightarrow \Delta\phi = -h \Delta\theta \quad \text{with} \quad \theta = \int \omega \, dt$$

particle ahead arrives earlier
 \Rightarrow smaller RF phase

For a given particle with respect to the reference one:

$$\Delta\omega = \frac{d}{dt}(\Delta\theta) = -\frac{1}{h} \frac{d}{dt}(\Delta\phi) = -\frac{1}{h} \frac{d\phi}{dt}$$

Since:

$$\eta = \frac{p_s}{\omega_{rs}} \left(\frac{d\omega}{dp} \right)_s$$

and

$$E^2 = E_0^2 + p^2 c^2$$

$$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets:

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h \eta \omega_{rs}} \frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h \eta \omega_{rs}} \dot{\phi}$$

Appendix: Second Energy-Phase Equation

The rate of energy gained by a particle is: $\frac{dE}{dt} = e\hat{V}\sin\phi \frac{\omega_r}{2\pi}$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left-hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}\Delta T_r + T_{rs}\Delta\dot{E} = \Delta E\dot{T}_r + T_{rs}\Delta\dot{E} = \frac{d}{dt}\left(T_{rs}\Delta E\right)$$

leads to the second energy-phase equation:

$$2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

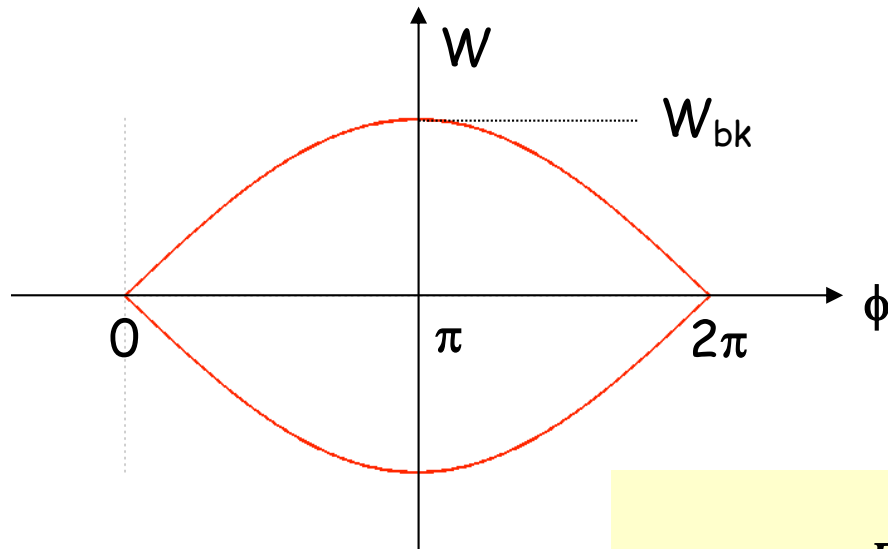
Appendix: Stationnary Bucket - Separatrix

This is the case $\sin\phi_s=0$ (no acceleration) which means $\phi_s=0$ or π . The equation of the separatrix for $\phi_s=\pi$ (above transition) becomes:

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos\phi = \Omega_s^2$$

$$\frac{\dot{\phi}^2}{2} = 2\Omega_s^2 \sin^2 \frac{\phi}{2}$$

Replacing the phase derivative by the (canonical) variable W :



with $C=2\pi R$

$$W = \frac{\Delta E}{\omega_0} = \frac{p_0 R}{h\eta\omega_0} \dot{\phi}$$

and introducing the expression for Ω_s leads to the following equation for the separatrix:

$$W = \pm \frac{R}{c} \sqrt{\frac{2q\hat{V}E_0}{\pi h\eta}} \sin \frac{\phi}{2} = \pm W_{bk} \sin \frac{\phi}{2}$$

Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos \phi_s} (\sin \phi - \sin \phi_s) = 0 \quad (\Omega_s \text{ as previously defined})$$

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = I$$

which for small amplitudes reduces to:

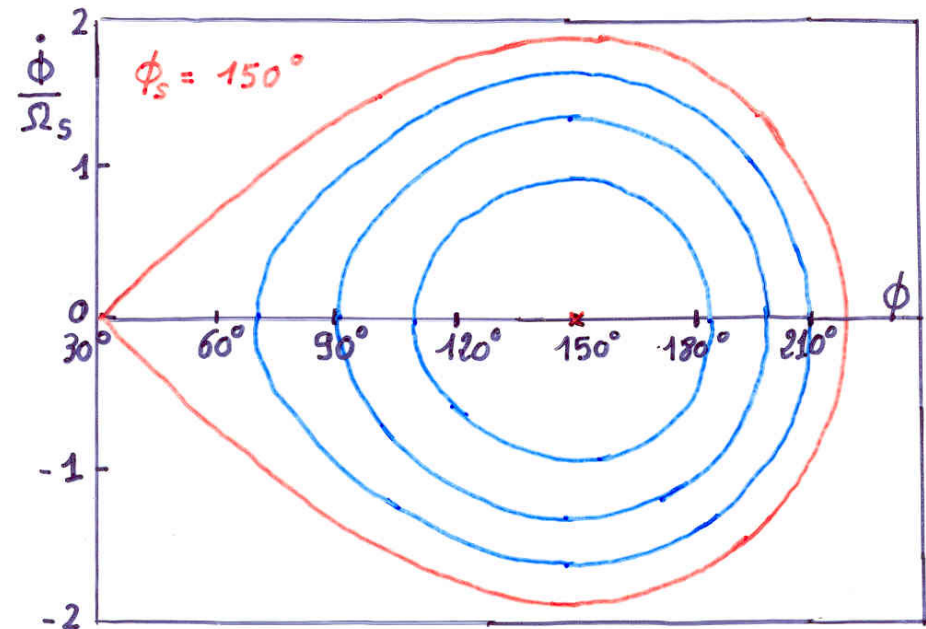
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \frac{(\Delta\phi)^2}{2} = I' \quad (\text{the variable is } \Delta\phi, \text{ and } \phi_s \text{ is constant})$$

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches $\pi - \phi_s$ the force goes to zero and beyond it becomes non restoring.

Hence $\pi - \phi_s$ is an extreme amplitude for a stable motion which in the phase space $(\frac{\dot{\phi}}{\Omega_s}, \Delta\phi)$ is shown as closed trajectories.



Equation of the **separatrix**:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos \phi_s} (\cos \phi + \phi \sin \phi_s) = -\frac{\Omega_s^2}{\cos \phi_s} (\cos(\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos \phi_m + \phi_m \sin \phi_s = \cos(\pi - \phi_s) + (\pi - \phi_s) \sin \phi_s$$

Energy Acceptance

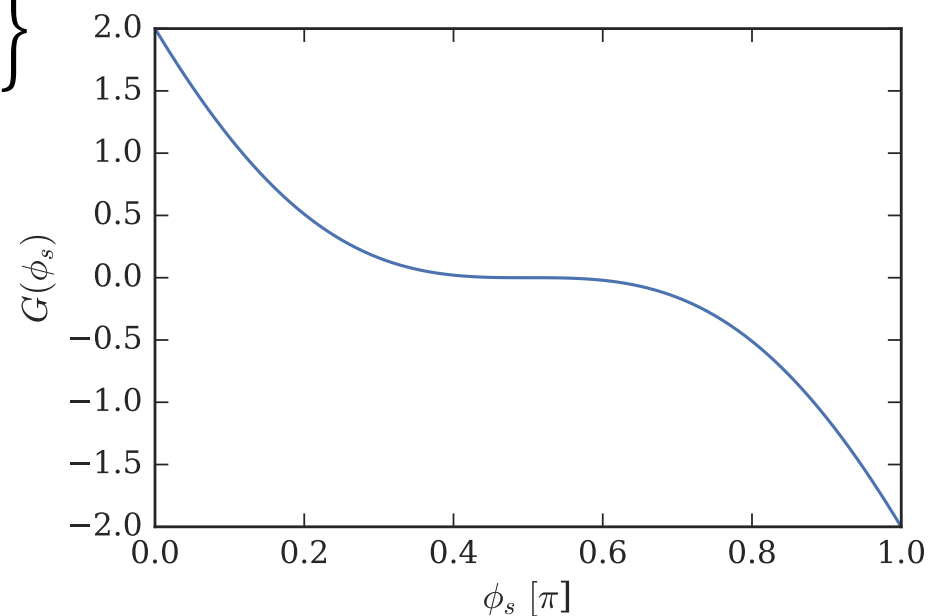
From the equation of motion it is seen that $\dot{\phi}$ reaches an extreme at $\phi = \phi_s$.
Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}_{\max}^2 = 2\Omega_s^2 \left\{ 2 + (2\phi_s - \pi) \tan \phi_s \right\}$$

That translates into an **energy acceptance**:

$$\left(\frac{\Delta E}{E_s} \right)_{\max} = \pm \beta \sqrt{\frac{e\hat{V}}{\pi h \eta E_s} G(\phi_s)}$$

$$G(\phi_s) = [2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s]$$



This “**RF acceptance**” depends strongly on ϕ_s and plays an important role for the capture at injection, and the stored beam lifetime.

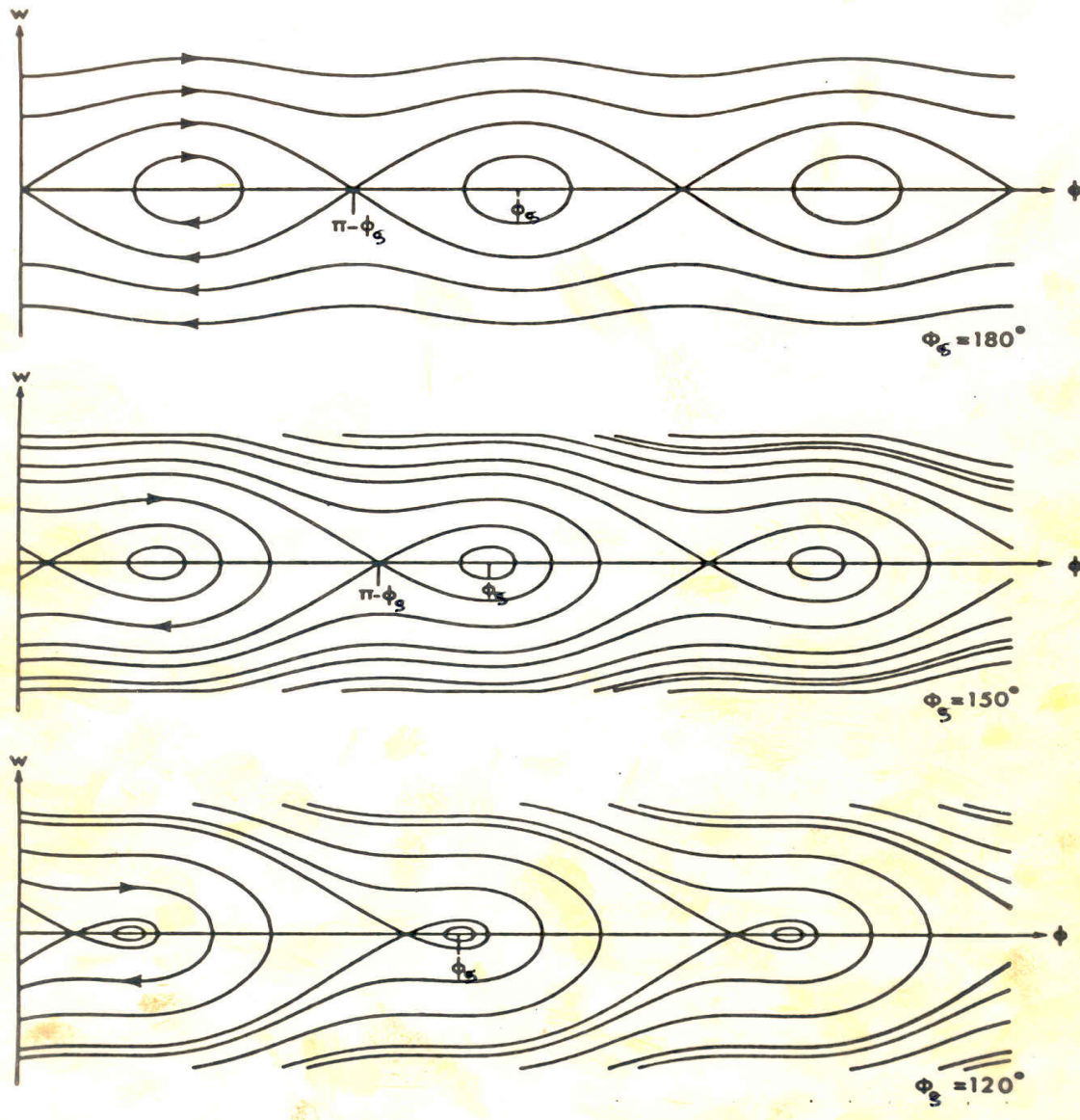
It's **largest** for $\phi_s=0$ and $\phi_s=\pi$ (**no acceleration**, depending on η).

It becomes smaller during acceleration, when ϕ_s is changing

Need a **higher RF voltage** for **higher acceptance**.

For the **same RF voltage** it is **smaller** for **higher harmonics** h .

RF Acceptance versus Synchronous Phase



The **areas of stable motion** (closed trajectories) are called "**BUCKET**". The number of circulating buckets is equal to " h ".

The phase extension of the **bucket is maximum** for $\phi_s = 180^\circ$ (or 0°) which means **no acceleration**.

During **acceleration**, the buckets get **smaller**, both in length and **energy acceptance**.

=> **Injection** preferably **without acceleration**.

Bucket height - bucket area

Setting $\phi=\pi$ in the previous equation gives the **height** of the **stationary bucket**:

$$W_{bk} = \frac{R}{c} \sqrt{\frac{2q\hat{V}E_0}{\pi h|\eta|}}$$

The **bucket area** is: $A_{bk} = 2 \int_0^{2\pi} W d\phi$

Since: $\int_0^{2\pi} \sin \frac{\phi}{2} d\phi = 4$

one gets:

$$A_{bk} = 8 W_{bk} = \frac{8R}{c} \sqrt{\frac{2q\hat{V}E_0}{\pi h|\eta|}}$$



$$W_{bk} = \frac{A_{bk}}{8}$$

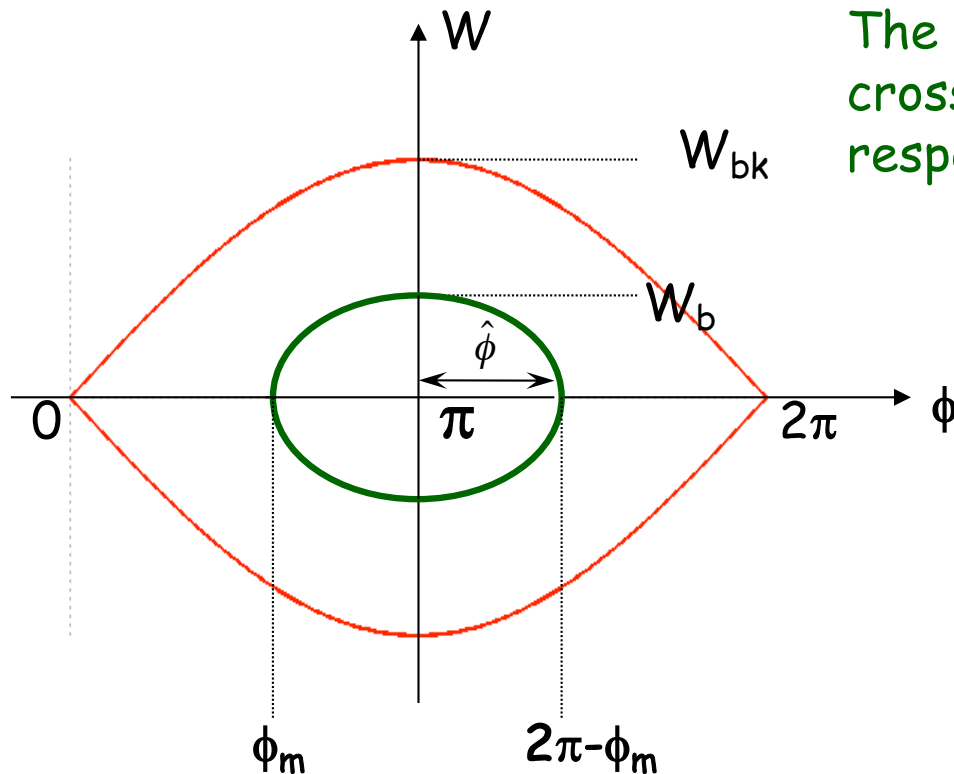
For an **accelerating bucket**, this **area** gets **reduced by** a factor depending on Φ_s :

$$\alpha(\phi_s) \approx \frac{1 - \sin \phi_s}{1 + \sin \phi_s}$$

Bunch Matching into a Stationary Bucket

A particle trajectory inside the separatrix is described by the equation:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s} (\cos\phi + \phi \sin\phi_s) = I \quad \xrightarrow{\phi_s = \pi} \quad \frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos\phi = I$$



The points where the trajectory crosses the axis are symmetric with respect to $\phi_s = \pi$

$$\frac{\dot{\phi}^2}{2} + \Omega_s^2 \cos\phi = \Omega_s^2 \cos\phi_m$$

$$\dot{\phi} = \pm \Omega_s \sqrt{2(\cos\phi_m - \cos\phi)}$$

$$W = \pm W_{bk} \sqrt{\cos^2 \frac{\varphi_m}{2} - \cos^2 \frac{\varphi}{2}}$$

$$\cos(\phi) = 2 \cos^2 \frac{\phi}{2} - 1$$

Bunch Matching into a Stationary Bucket (2)

Setting $\phi = \pi$ in the previous formula allows to calculate the bunch height:

$$W_b = W_{bk} \cos \frac{\phi_m}{2} = W_{bk} \sin \frac{\hat{\phi}}{2}$$

or:

$$W_b = \frac{A_{bk}}{8} \cos \frac{\phi_m}{2}$$

$$\longrightarrow \left(\frac{\Delta E}{E_s} \right)_b = \left(\frac{\Delta E}{E_s} \right)_{RF} \cos \frac{\phi_m}{2} = \left(\frac{\Delta E}{E_s} \right)_{RF} \sin \frac{\hat{\phi}}{2}$$

This formula shows that for a given bunch energy spread the proper matching of a **shorter bunch** (ϕ_m close to π , $\hat{\phi}$ small) will **require** a bigger RF acceptance, hence a **higher voltage**

For small oscillation amplitudes the equation of the ellipse reduces to:

$$W = \frac{A_{bk}}{16} \sqrt{\hat{\phi}^2 - (\Delta\phi)^2} \longrightarrow \left(\frac{16W}{A_{bk}\hat{\phi}} \right)^2 + \left(\frac{\Delta\phi}{\hat{\phi}} \right)^2 = 1$$

Ellipse area is called longitudinal emittance

$$A_b = \frac{\pi}{16} A_{bk} \hat{\phi}^2$$

Longitudinal Dynamics - Electrons

At **relativistic velocity phase oscillations stop** - the bunch is frozen longitudinally.
 \Rightarrow Acceleration can be at the crest of the RF for maximum energy gain.

Electrons injected into a TW structure designed for $v=c$:

- \rightarrow at $v=c$ remain at the injection phase.
- \rightarrow at $v < c$ will move from injection phase ϕ_0 to an asymptotic phase ϕ , which depends on gradient E_0 and β_0 at injection.

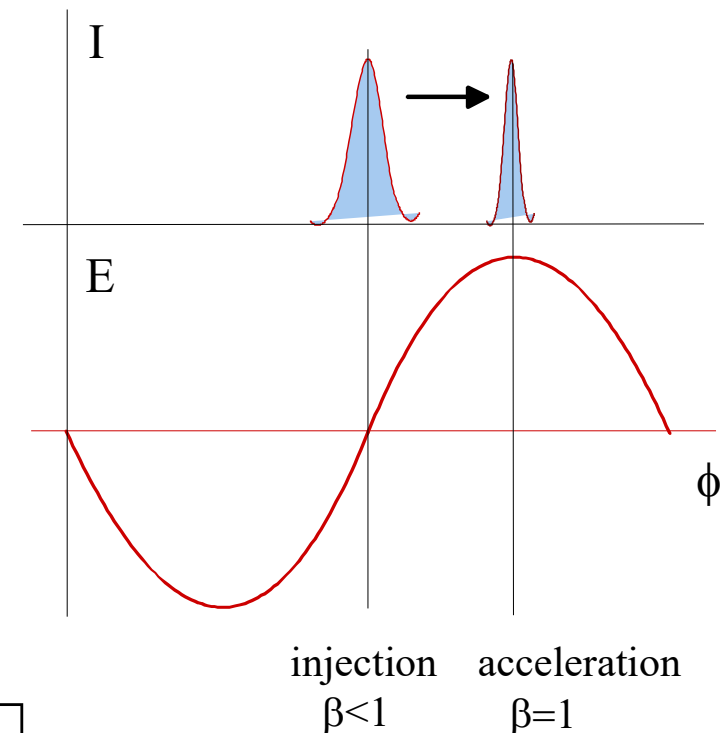
The beam can be injected with an offset in phase, to reach the crest of the wave at $\beta=1$

Capture condition, relating gradient E_0 and β_0 :

$$E_0 \geq \frac{2\pi}{\lambda_g} \frac{mc^2}{q} \left[\sqrt{\frac{1-\beta_0}{1+\beta_0}} \right]$$

Example: $\lambda=10\text{cm} \rightarrow W_{\text{in}}=150\text{ keV}$ for $E_0=8\text{ MV/m}$.

$$\cos \phi = \cos \phi_0 + \frac{2\pi}{\lambda_g} \frac{mc^2}{qE_0} \left[\sqrt{\frac{1-\beta}{1+\beta}} - \sqrt{\frac{1-\beta_0}{1+\beta_0}} \right]$$

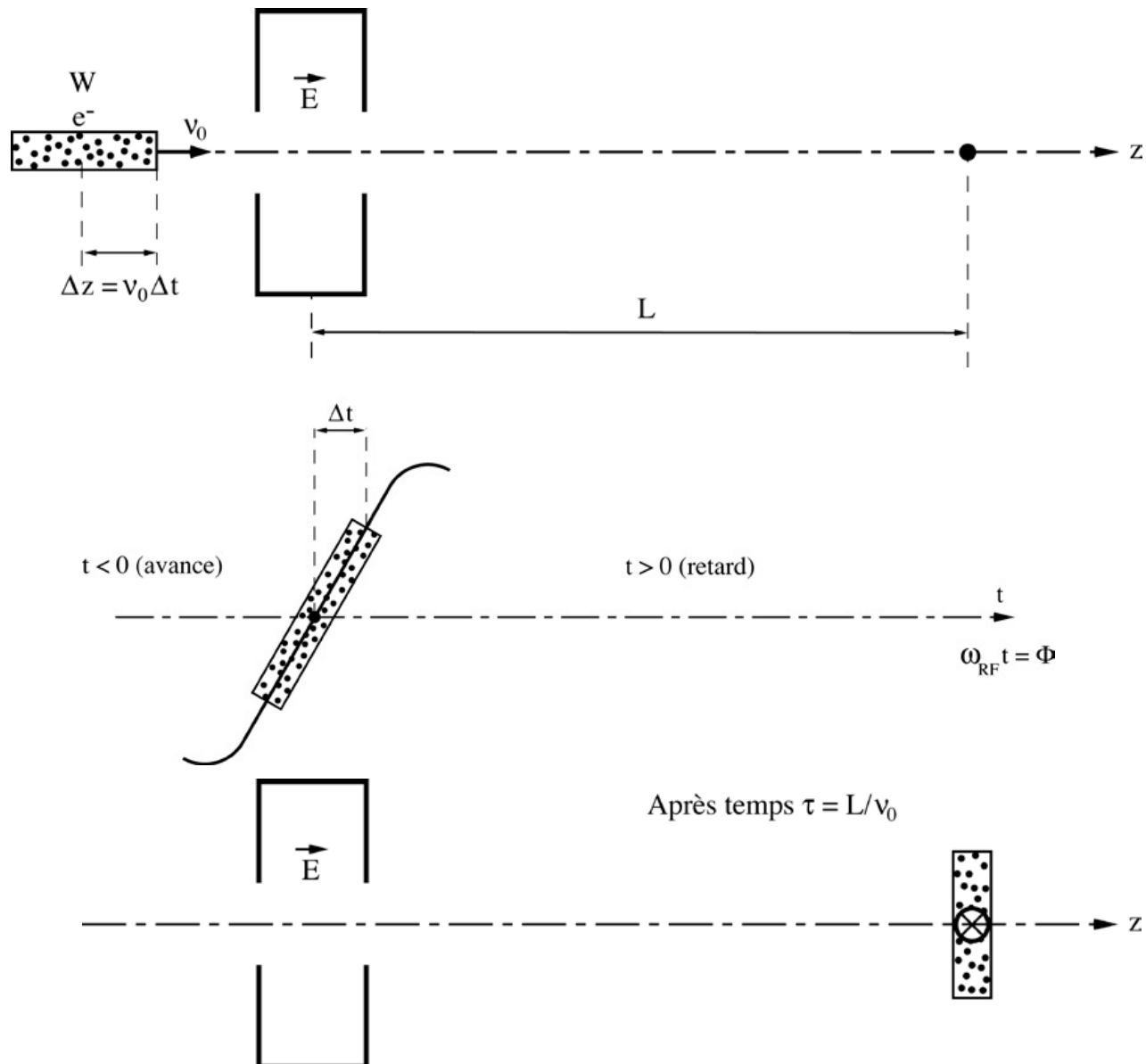


In high current linacs, a bunching and pre-acceleration section up to 4-10 MeV prepares the injection in the TW structure (that occurs already on the crest)

Bunching with a Pre-buncher

A long bunch coming from the gun enters an RF cavity. The reference particle is the one which has no velocity change. The others get accelerated or decelerated, so the bunch gets an energy and velocity modulation.

After a distance L bunch gets shorter: **bunching effect**. This short bunch can now be captured more efficiently by a TW structure ($v_\phi = c$).



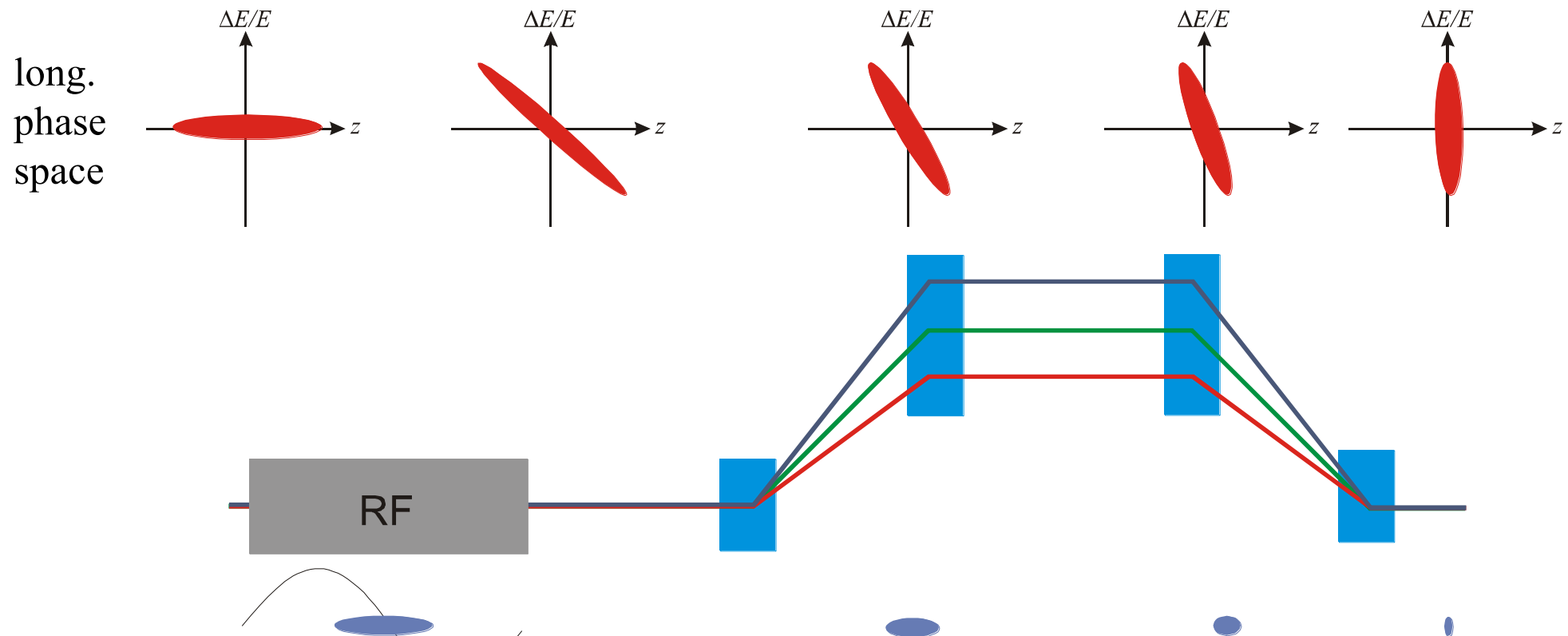
Bunch compression

At ultra-relativistic energies ($\gamma \gg 1$) the longitudinal motion is frozen.
For linear e^+/e^- colliders, you need very short bunches (few $100\text{-}50\mu\text{m}$).

Solution: introduce **energy/time correlation** + a magnetic **chicane**.

Increases energy spread in the bunch \Rightarrow chromatic effects

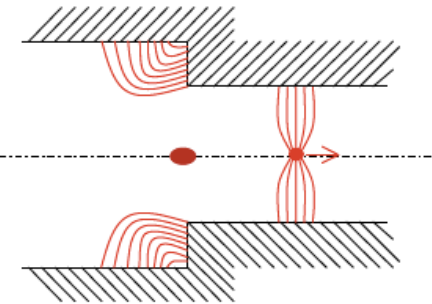
\Rightarrow **compress at low energy** before further acceleration to reduce relative $\Delta E/E$



Longitudinal Wake Fields - Beamloading

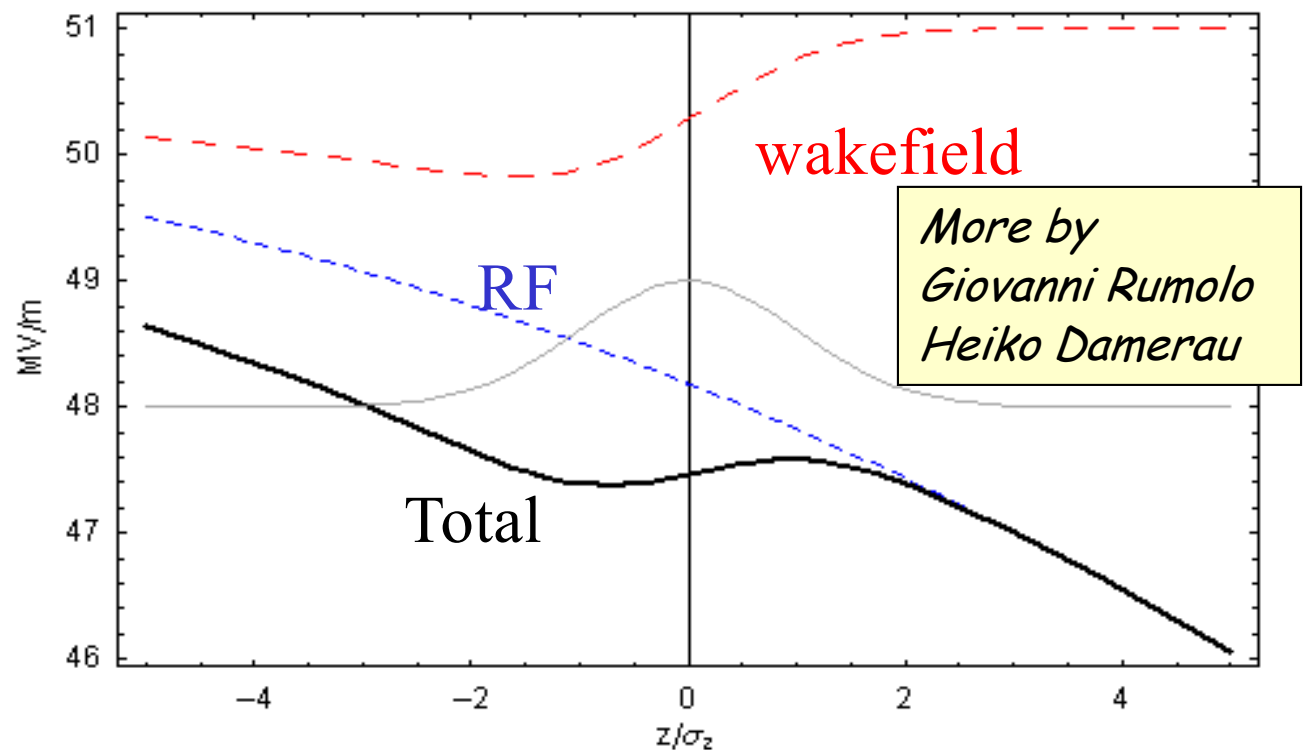
Beam induces wake fields in cavities (in general when chamber profile changing)
⇒ **decreasing RF field** in cavities
(beam absorbs RF power when accelerated)

Particles within a bunch see a decreasing field
⇒ energy gain different **within** the single bunch



Locating bunch **off-crest**
at the best RF phase
minimises energy spread

Example: Energy gain
along the bunch
in the NLC linac (TW):



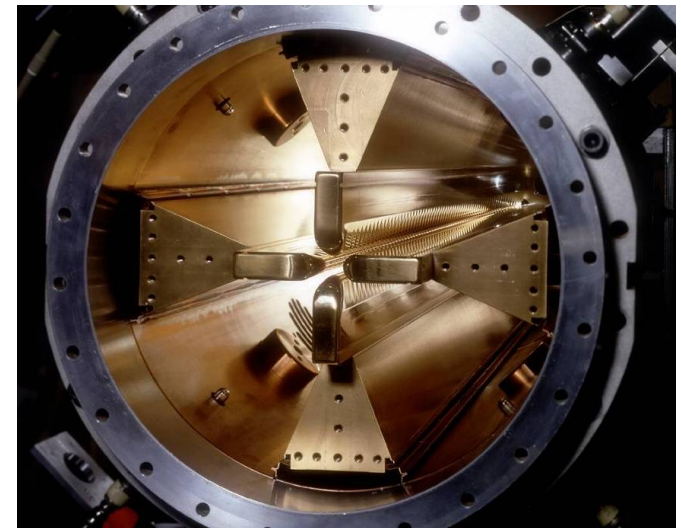
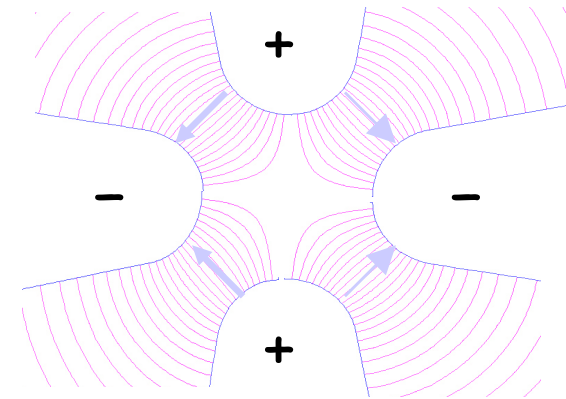
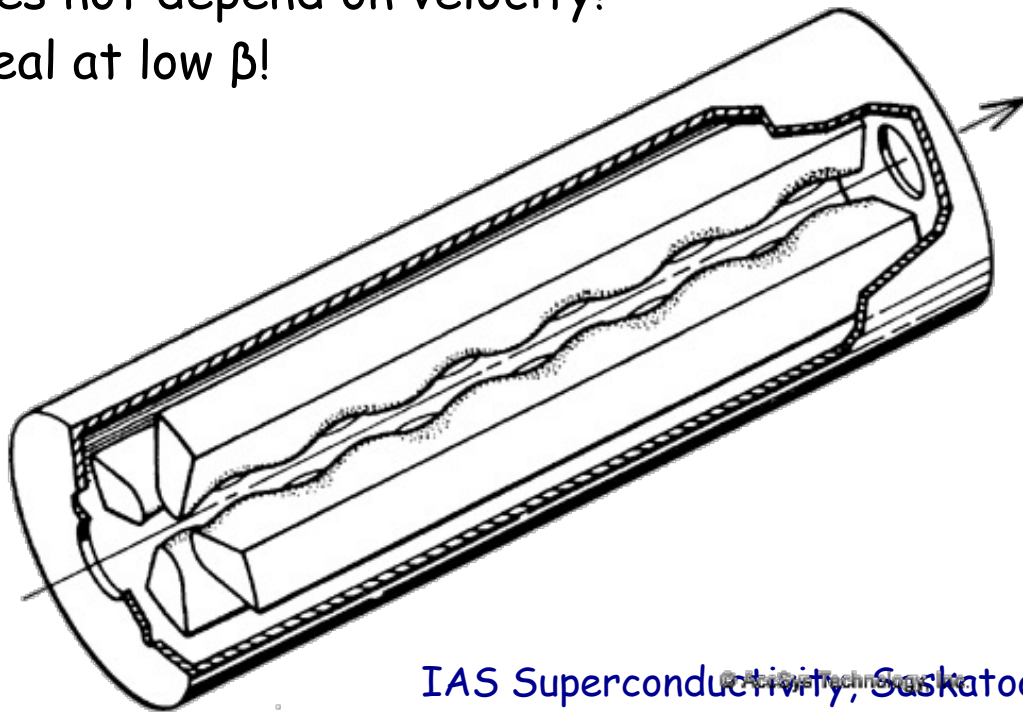
The Radio-Frequency Quadrupole - RFQ

Initial acceleration difficult for protons and ions at low energy
(space charge, low $\beta \Rightarrow$ short cell dimensions, bunching needed)

RFQ = Electric quadrupole

focusing channel + bunching + acceleration

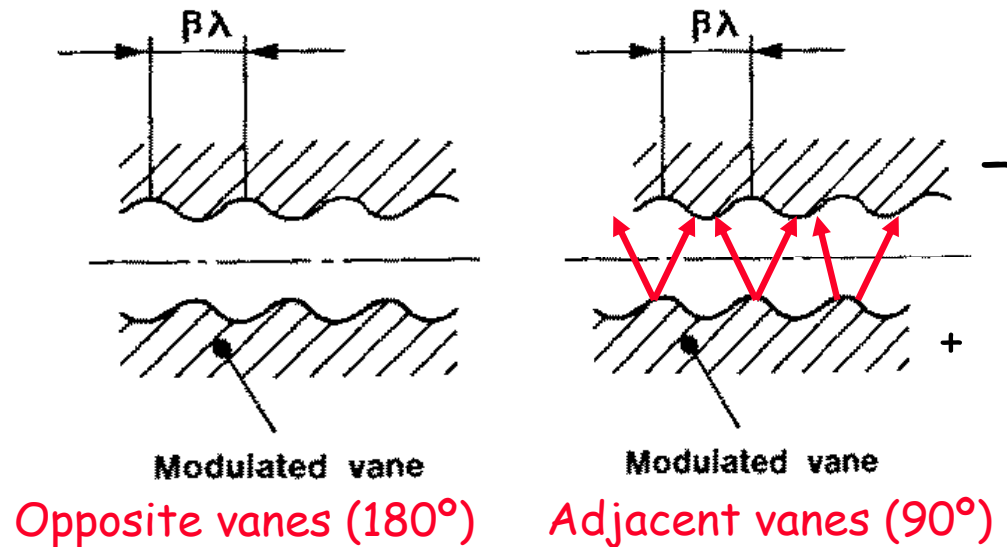
Alternating electric quadrupole field gives
transverse focusing like magnetic focusing channel.
Does not depend on velocity!
Ideal at low β !



The Radio-Frequency Quadrupole - RFQ

The vanes have a longitudinal modulation with period = $\beta\lambda$
 → this creates a longitudinal component of the electric field.

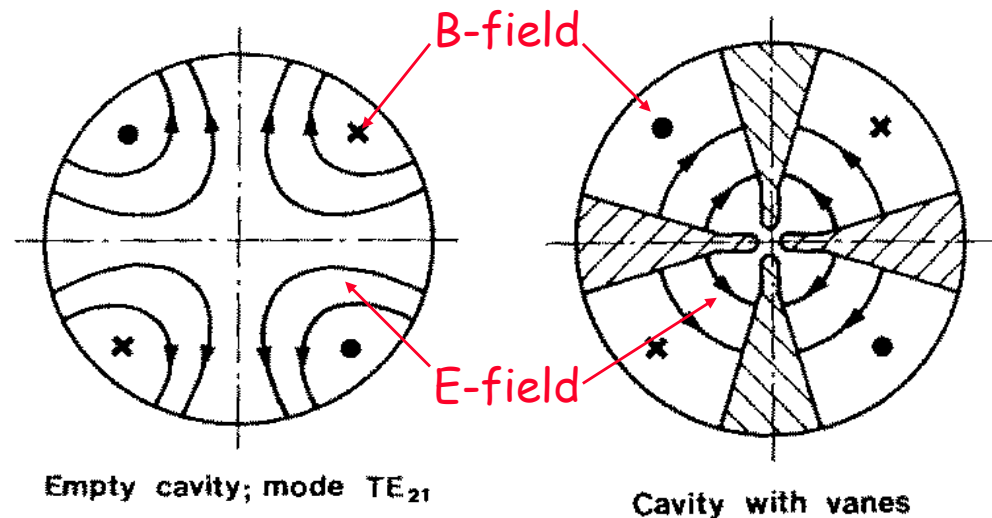
The modulation corresponds exactly to a series of RF gaps and can provide acceleration.



RF Field excitation:

An empty cylindrical cavity can be excited on different modes.

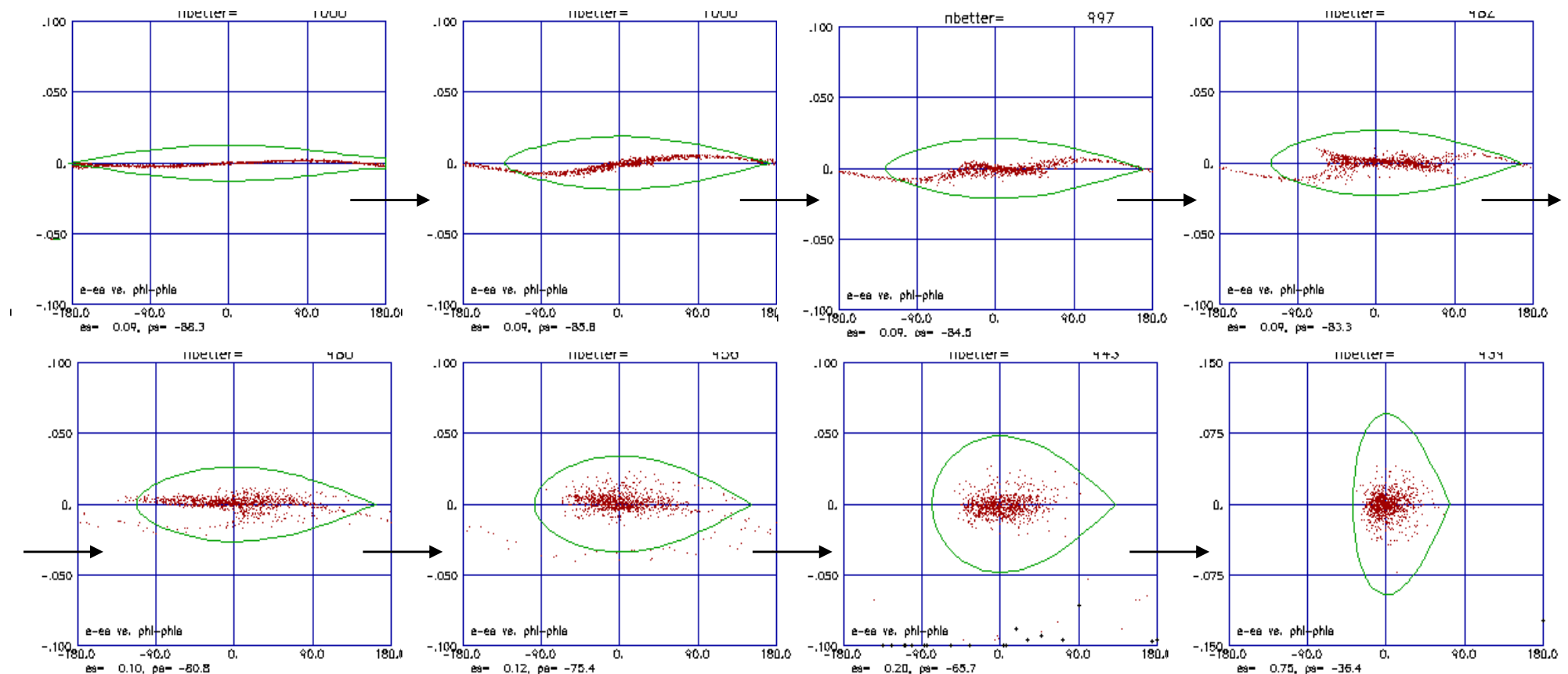
Some of these modes have only transverse electric field (the TE modes), and one uses in particular the "quadrupole" mode, the TE_{210} .



RFQ Design + Longitudinal Phase Space

RFQ design: The modulation period can be slightly adjusted to change the phase of the beam inside the RFQ cells, and the amplitude of the modulation can be changed to change the accelerating gradient

→ start with some bunching cells, progressively bunch the beam (adiabatic bunching channel), and only in the last cells accelerate.



Longitudinal beam profile of a proton beam along the CERN RFQ2

IAS Superconductivity, Saskatoon, Canada, July 2023